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Tools for analyzing the intersection curve between two quadrics through projection and lifting

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ABSTRACT

This article introduces several efficient and easy-to-use tools to analyze the intersection curve between two quadrics, on the basis of the study of its projection on a plane (the so-called cutcurve) to perform the corresponding lifting correctly. This approach is based on an efficient way of determining the topology of the cutcurve through only solving one degree eight (at most) univariate equation and several quadratic univariate equations, intersecting two pairs of conics and, when the parameterization of the cutcurve in closed form cannot be determined, computing the real roots of several degree four univariate squarefree polynomials whose number (of real roots) is known in advance.

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1. Introduction

Computing the intersection curve of two surfaces is a central problem in many areas, such as the CAD/CAM treatment of complicated shapes, the design of three-dimensional (3D) objects, computer animation, NC machining and the creation of boundary representation in solid modeling. For surfaces with a prescribed structure (algebraic or geometric), extensive literature has described algorithms for solving the surface intersection problem in a very efficient manner by using the particular structure of the considered surfaces, such as ruled, revolution, canal or torus (see for example [1–3]).

Quadrics are the simplest curved surfaces used in many applications, and computing their intersection is a practically relevant problem to solve. Many applications of intersection queries involving two quadrics can be found in different fields of science and technology, such as global navigation satellite system modeling [4], computer aided design [5], optical engineering [6], computer vision [7], robotics [8] and statistical physics [9].

Algorithms addressing this problem by using floating point arithmetic techniques are sensitive to rounding errors, thus achieving a low running time to the detriment of their correctness. However, using symbolic methods ensures the correctness of the results, because these methods are based on exact arithmetic (if the considered quadrics are defined in exact terms). However, their performance is typically significantly lower than that of methods based on numerical techniques.

Levin [10,11] developed a method to parameterize the intersection curve of two quadrics according to the analysis of the pencil that they generate. Wilf et al. [12] have improved on Levin's ruled-surface parameterization scheme. However, Levin's method often fails to find the intersection curve when it is singular and generates a parameterization that involves the square root of some polynomial [13]. Moreover, in working with floating point numbers, Levin's method sometimes

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outputs results that are topologically incorrect and even fail to produce any parameterization [13]. Wang et al. [14] have reduced the computation of the intersection curve to the analysis of plane cubic curves. Farouki et al. [15] have performed a comprehensive study of the degenerated cases of quadrics intersection by using factorization of multivariate polynomials and Segre characteristics. This method shows the exact parameterization of the intersection curve in many cases.

Later, Wang et al. [16] improved Levin's method, making it capable of computing geometric and structural information, i.e., irreducibility, singularities and the number of connected components. Dupont et al. [13,17,18] have presented a near-optimal algorithm for computing the explicit representation of the intersection curve of two arbitrary quadrics in the projective space, whose coefficients are rational numbers, by using the reduction of quadratic forms, thus producing new results fully characterizing and computing the intersection curve of two quadrics. The performance of the implementation of this algorithm has been analyzed in [19].

Others have restricted the type of quadrics considered and have defined specific procedures for each case [20–24], taking advantage of the fact that a geometric approach is typically more stable than an algebraic or numerical approach [13]. However, these approaches are limited to planar intersections and natural quadrics. Mourrain et al. [25] have proposed an algorithm that reduces the intersection of two quadrics to a dynamic two-dimensional problem.

One possible strategy for computing the intersection curve between two quadrics in 3D space is based on analyzing its projection onto one plane [26]. The idea underlying this method is reducing the initial 3D problem to the computation of the arrangement of three real algebraic plane curves defined implicitly. After determination and analysis of the projection of the intersection curve onto a plane, the intersection curve between the quadrics can be recovered by determining the lifting of the projection curve. Implementation and theoretical aspects of this approach can also be found in [27] and [28].

Here, we introduce several, easy to use, tools to analyze the intersection curve between two quadrics through projection onto a plane and lifting. In some cases, the exact parameterization of the intersection curve (involving radicals if needed) can be determined, and in other cases, the output (topologically correct) is presented as the lifting of the discretization of the branches of the projection curve after its topology has been fully determined. The main novelty of this approach is the detailed analysis of the projection curve: because it is a very special quartic curve, its singular and critical points can easily be determined (through solving one degree eight, at most, univariate equation and several quartic and quadratic univariate equations); its topology is thus determined in a very efficient manner.

This paper is organized as follows. Section 2 reviews how conics and quadrics are represented, and briefly presents resultants and subresultants for the sake of completeness. Resultants are used in Section 3 to characterize the projection of the intersection curve (called the cutcurve herein) by using a bivariate polynomial of degree at most four and two bivariate polynomial inequalities of degree at most two. Our approach is based on a detailed analysis of the topology of the cutcurve, following [27,28], which is included in Section 4. We additionally show how to lift the cutcurve to the intersection curve between the two considered quadrics. Some examples are given in Section 5, in which the results of the implementation in `Maple` are also reported. Conclusions are presented in Section 6.

To simplify the description of the algorithm and to avoid a long case-by-case analysis, we will examine only quadrics whose defining equations have the forms $z^2 + p_1(x, y)z + p_0(x, y)$ and $z^2 + q_1(x, y)z + q_0(x, y)$. The general case is very easy to derive from the descriptions herein.

2. Preliminaries: quadrics, conics and subresultants

In this section, first, we introduce how quadrics will be represented in computing the intersection curve. Because we will project the considered quadrics onto the xy plane, and the boundary of this region will be a finite number of conic arcs, we introduce how these regions will be represented and manipulated. Then we provide a brief introduction to resultants and subresultants and how they will be applied later to study the intersection curve between two quadrics.

2.1. Representing quadrics (and conics)

This section introduces how quadrics will be represented when computing the intersection curve. Quadrics are surfaces defined by degree two polynomials in x, y and z. The equation of any quadric $\mathcal{A}$ in $\mathbb{R}^3$ can be written as

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0$$

or in matricial form $\begin{pmatrix} x & y & z & 1 \end{pmatrix} A \begin{pmatrix} x & y & z & 1 \end{pmatrix}^T = 0$ where A is the symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}.$$

For conics, the treatment is analogous (and standard). Conics will be used later to define the boundary of regions in the plane: more precisely, the regions defined by the projection of the two considered quadrics and where the projection of their intersection curve is to be found. Subsequently, we assume that determining the intersection points of two conics and manipulating the regions of the plane defined by two inequalities involving degree two or one polynomials are easily performed (see [29,30]).

2.2. Resultants and subresultants

Resultants and subresultants are the algebraic tools used to determine both the projection of the intersection curve between the considered quadrics and its lifting from the plane to the 3D space, because they provide a very easy and compact way of characterizing the greatest common divisor of two polynomials ($f(x, y, z)$ and $g(x, y, z)$ in our case) when they involve parameters (x and y in our case, because we will eliminate z).

Definition 2.1. Let

$$P(T) = \sum_{i=0}^{m} a_{m-i}T^i \qquad \text{and} \qquad Q(T) = \sum_{i=0}^{n} b_{n-i}T^i$$

be two polynomials with coefficients in a field ($\mathbb{Q}$ or $\mathbb{R}$ in our case). We define the j-th subresultant polynomial of P and Q with respect to T as follows (as in [31]):

$$\mathbf{Sres}_j(P, Q; T) = \begin{array}{c} \left|\begin{array}{ccccccccc} a_0 & a_1 & a_2 & \dots & \dots & a_m & & & \\ & \ddots & \ddots & \ddots & & & \ddots & & \\ & & a_0 & a_1 & a_2 & \dots & \dots & a_m & \\ & & & & & 1 & -T & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & 1 & -T \\ b_0 & b_1 & b_2 & \dots & \dots & \dots & b_n & & \\ & \ddots & \ddots & \ddots & & & & \ddots & \\ & & b_0 & b_1 & b_2 & \dots & \dots & \dots & b_n \end{array}\right| \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\\a\\a\\a\\a\\a\end{array}}\right\} n-j \\ \left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} j \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} m-j \end{array}$$

and we define the j-th subresultant coefficient of P and Q with respect to T, $\mathbf{sres}_j(P, Q; T)$, as the coefficient of T^j in $\mathbf{Sres}_j(P, Q; T)$. The resultant of P and Q with respect to T is:

$$\mathbf{Resultant}(P, Q; T) = \mathbf{Sres}_0(P, Q; T) = \mathbf{sres}_0(P, Q; T) \quad .$$

There are many ways of defining and computing subresultants: for a short introduction, see [32] and the references cited therein. Subresultants allow for easy characterization of the degree of the greatest common divisor of two univariate polynomials whose coefficients depend on one or several parameters. Because the resultant of P and Q is equal to the polynomial $\mathbf{sres}_0(P, Q; T)$, we have:

$$\mathbf{sres}_0(P, Q; T) = 0 \text{ if and only if there exists } T_0 \text{ such that } P(T_0) = 0 \text{ and } Q(T_0) = 0. \tag{1}$$

More generally, the determinants $\mathbf{sres}_j(P, Q; T)$, which are the formal leading coefficients of the subresultant sequence for P and Q, can be used to compute the greatest common divisor of P and Q, owing to the following equivalence:

$$\mathbf{Sres}_i(P, Q; T) = \gcd(P, Q) \Longleftrightarrow \left\{ \begin{array}{c} \mathbf{sres}_0(P, Q; T) = \cdots = \mathbf{sres}_{i-1}(P, Q; T) = 0 \\ \mathbf{sres}_i(P, Q; T) \neq 0 \end{array} \right. \tag{2}$$

Let f and g be the two polynomials in $\mathbb{R}[x, y, z]$

$$f(x, y, z) = z^2 + p_1(x, y)z + p_0(x, y) \qquad g(x, y, z) = z^2 + q_1(x, y)z + q_0(x, y)$$

($\deg(p_1(x, y)) \leq 1$, $\deg(p_0(x, y)) \leq 2$, $\deg(q_1(x, y)) \leq 1$ and $\deg(q_0(x, y)) \leq 2$) defining the two quadrics whose intersection curve is to be computed. Then the resultant of f and g, with respect to z, is equal to:

$$\begin{aligned} \mathbf{S}_0(x, y) &\stackrel{\text{def}}{=} \mathbf{Resultant}(f, g; z) = \left|\begin{array}{cccc} 1 & p_1(x, y) & p_0(x, y) & 0 \\ 0 & 1 & p_1(x, y) & p_0(x, y) \\ 1 & q_1(x, y) & q_0(x, y) & 0 \\ 0 & 1 & q_1(x, y) & q_0(x, y) \end{array}\right| = \\ &= (p_0(x, y) - q_0(x, y))^2 - (p_1(x, y) - q_1(x, y)) \left|\begin{array}{cc} p_0(x, y) & p_1(x, y) \\ q_0(x, y) & q_1(x, y) \end{array}\right|. \end{aligned} \tag{3}$$

The total degree of $\mathbf{S}_0(x, y)$ is at most four.

Computing the intersection of the two quadrics defined by f and g is equivalent to solving in $\mathbb{R}^3$ the polynomial system of equations

$$f(x, y, z) = 0, \qquad g(x, y, z) = 0.$$

The solution set to be computed, when non-empty, may include curves and isolated points. We will use the above polynomial system of equations, which under some conditions is equivalent to

$$\mathbf{S}_0(x, y) = 0, \qquad (q_1(x, y) - p_1(x, y))z + (q_0(x, y) - p_0(x, y)) = 0.$$

Analysis of $\mathbf{S}_0(x, y) = 0$ in $\mathbb{R}^2$ will be called the projection step, and moving the information obtained in $\mathbb{R}^2$ to $\mathbb{R}^3$ will be called the lifting step. This is the customary terminology used when computing the cylindrical algebraic decomposition of a finite set of multivariate polynomials (see [33], for example).

3. Projecting the intersection curve: the cutcurve

In this section, we characterize the projection of the intersection curve of two quadrics onto the (x, y)-plane. The usual method of handling projections of algebraic sets involves tools from elimination theory. Because we project from $\mathbb{R}^3$ onto $\mathbb{R}^2$, this description also involves polynomial inequalities.

Let f and g be the two polynomials in $\mathbb{R}[x, y, z]$

$$f(x, y, z) = z^2 + p_1(x, y)z + p_0(x, y) \qquad g(x, y, z) = z^2 + q_1(x, y)z + q_0(x, y) \tag{4}$$

with $\deg(p_1) \leq 1$, $\deg(p_0) \leq 2$, $\deg(q_1) \leq 1$ and $\deg(q_0) \leq 2$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the corresponding quadrics:

$$\mathcal{E}_1 : \left\{(x, y, z) \in \mathbb{R}^3 : f\,(x, y, z) = 0\right\} \qquad \mathcal{E}_2 : \left\{(x, y, z) \in \mathbb{R}^3 : g\,(x, y, z) = 0\right\},$$

and Π be the projection:

$$\begin{array}{cccc} \Pi : & \mathbb{R}^3 & \rightarrow & \mathbb{R}^2 \\ & (x, y, z) & \mapsto & (x, y) \end{array}$$

Let $\Delta_{\mathcal{E}_1}(x, y) = p_1(x, y)^2 - 4p_0(x, y)$ and $\Delta_{\mathcal{E}_2}(x, y) = q_1(x, y)^2 - 4q_0(x, y)$ be the discriminants of $f(x, y, z)$ and $g(x, y, z)$ (respectively) with respect to z.

Next, an easily proven theorem characterizes the set $\Pi\,(\mathcal{E}_1 \cap \mathcal{E}_2)$, the projection of the intersection curve of $\mathcal{E}_1$ and $\mathcal{E}_2$.

Theorem 3.1.

$$\Pi\,(\mathcal{E}_1 \cap \mathcal{E}_2) = \left\{(x, y) \in \mathbb{R}^2 : \mathbf{S}_0(x, y) = 0,\, \Delta_{\mathcal{E}_1}(x, y) \geq 0,\, \Delta_{\mathcal{E}_2}(x, y) \geq 0\right\}.$$

Proof. If $(a, b) \in \Pi\,(\mathcal{E}_1 \cap \mathcal{E}_2)$, then there exists $c \in \mathbb{R}$ such that $(a, b, c) \in \mathcal{E}_1 \cap \mathcal{E}_2$. Thus $f(a, b, c) = 0$ and $g(a, b, c) = 0$, $f(a, b, z)$ and $g(a, b, z)$ have a common root ($c \in \mathbb{R}$), and we conclude that $\mathbf{S}_0(a, b) = 0$. Because c is a real root of $f(a, b, z)$ and $g(a, b, z)$, we also have $\Delta_{\mathcal{E}_1}(a, b) \geq 0$ and $\Delta_{\mathcal{E}_2}(a, b) \geq 0$.

However, if $(a, b) \in \mathbb{R}^2$ verifies $\mathbf{S}_0(a, b) = 0$, then $f(a, b, z)$ and $g(a, b, z)$ have a common root $c \in \mathbb{C}$: $f(a, b, c) = 0$ and $g(a, b, c) = 0$ (according to (2)). However, if $\Delta_{\mathcal{E}_1}(a, b) \geq 0$ and $\Delta_{\mathcal{E}_2}(a, b) \geq 0$, then c must be a real solution of $f(a, b, z) = 0$ and $g(a, b, z) = 0$. Therefore, $(a, b) \in \Pi\,(\mathcal{E}_1 \cap \mathcal{E}_2)$. $\square$

The previous theorem provides a precise description of the projection of the intersection curve of two quadrics when their defining equations have the structure introduced in (4). This projection corresponds to the part of the curve

$$\{(x, y) \in \mathbb{R}^2 \colon \mathbf{S}_0(x, y) = 0\}$$

inside the region

$$\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2} \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{R}^2 \colon \Delta_{\mathcal{E}_1}(x, y) \geq 0,\, \Delta_{\mathcal{E}_2}(x, y) \geq 0\}.$$

Thus, the region where the projection of the intersection curve of the two considered quadrics is found is bounded by a finite set of conic arcs, because any $\Delta_{\mathcal{E}_i}(x, y)$ is a polynomial in $\mathbb{R}[x, y]$ of total degree bounded by 2.

In [27], the curve in $\mathbb{R}^2$ defined by $\mathbf{S}_0(x, y) = 0$ is called the cutcurve of $\mathcal{E}_1$ and $\mathcal{E}_2$, and the curve in $\mathbb{R}^2$ defined by $\Delta_{\mathcal{E}_i}(x, y) = 0$ is called the silhouette of $\mathcal{E}_i$. We slightly modify the cutcurve definition to make it more suitable for our purposes.

Definition 3.2. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two quadrics in $\mathbb{R}^3$. The cutcurve of $\mathcal{E}_1$ and $\mathcal{E}_2$ is the set

$$\Pi\,(\mathcal{E}_1 \cap \mathcal{E}_2) = \left\{(x, y) \in \mathbb{R}^2 : \mathbf{S}_0(x, y) = 0,\, \Delta_{\mathcal{E}_1}(x, y) \geq 0,\, \Delta_{\mathcal{E}_2}(x, y) \geq 0\right\}.$$

According to Theorem 3.1, the cutcurve of $\mathcal{E}_1$ and $\mathcal{E}_2$ is equal to the projection of $\mathcal{E}_1 \cap \mathcal{E}_2$ onto the xy plane, $\Pi(\mathcal{E}_1 \cap \mathcal{E}_2)$. The cutcurve of $\mathcal{E}_1$ and $\mathcal{E}_2$ can be a curve, part of a curve (i.e., a semialgebraic set) or even a single point but always a semialgebraic set. We refer to the singular points of the cutcurve as those points that simultaneously vanish both partial derivatives of $\mathbf{S}_0(x, y)$. The whole cutcurve (or a part of it) may be singular (see Examples 3.3 and 5.2).

Example 3.3. Let f and g be the polynomials

$$f(x, y, z) = z^2 + x^2 + y^2 - 7 \qquad g(x, y, z) = z^2 - x^2 + xy + 2x - y^2$$

defining the two quadrics $\mathcal{E}_1$ and $\mathcal{E}_2$ whose intersection curve is to be computed. In this case, we have:

$$\mathbf{S}_0(x, y) = (-2x^2 + xy - 2y^2 + 2x + 7)^2,$$

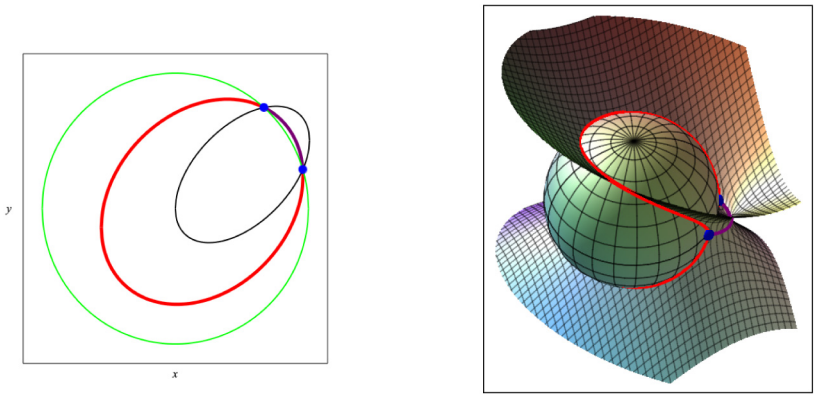

Fig. 1. Left: silhouette curves (green and black), cutcurve (red) and the part of the curve $\mathbf{S}_0(x, y) = 0$ that cannot be lifted (purple). Right: lifting of the intersection points (blue) between the silhouette curves and the cutcurve and the intersection curve (red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\Delta_{\mathcal{E}_1}(x, y) = -4x^2 - 4y^2 + 28, \qquad \Delta_{\mathcal{E}_2}(x, y) = 4x^2 - 4xy + 4y^2 - 8x$$

and

$$\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2} = \{(x, y) \in \mathbb{R}^2 : -x^2 - y^2 + 7 \geq 0,\ x^2 - xy + y^2 - 2x \geq 0\}.$$

In this case, the curve in $\mathbb{R}^2$ defined by $\mathbf{S}_0(x, y) = 0$ is not completely contained in $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$. The cutcurve is equal to the portion of the ellipse $-2x^2 + xy - 2y^2 + 2x + 7 = 0$ inside the circle $x^2 + y^2 \leq 7$ (see Fig. 1, left), and the whole cutcurve is singular.

In this case, we can parameterize the cutcurve either by solving y in terms of x in $-2x^2 + xy - 2y^2 + 2x + 7 = 0$ or by rationally parameterizing this ellipse. In both cases, knowing where the cutcurve intersects the silhouette curves is important to determine the correct portions of the ellipse to lift to obtain the intersection curve between the considered quadrics.

The method that we propose for analyzing the intersection curve between two quadrics is based on the analysis of the cutcurve and its lifting. When the cutcurve is easy, as in the previous example (a conic, a product of lines, a product of conics, etc.), we propose to directly parameterize the cutcurve and then carefully lift it to $\mathbb{R}^3$. As in the previous example, this lifting requires more information, for example, the parts of $\mathbf{S}_0(x, y) = 0$ that must be taken into account to obtain the intersection curve. The next section provides a detailed study of the cutcurve to be used in both cases: either when the cutcurve can be parameterized rationally or involves square roots, or when this is not the case.

4. Analyzing the cutcurve

In this section, we study the cutcurve of the two quadrics whose intersection curve is to be computed; this is achieved by computing its topology. The usual method of computing the topology of a real algebraic plane curve defined implicitly requires computing its critical points (those that are singular or with a vertical tangent), regular points sharing the same projection as one of the critical points and the number of branches connecting all these points (connecting branches and points will be particularly easy in our case, because we are working with a very special quartic): for details see [33,34] and Section 4.7 (especially Fig. 3).

In our case, because we are working with the cutcurve, we need to restrict our attention to the admissible region; this is why we will also pay special attention to the intersection points between the cutcurve and the silhouette curves. These points will include those points where the different cutcurve components stop and start again (unless both curves are tangent). Fig. 1 (left) shows one concrete example of this situation.

The analysis of the cutcurve is guided by the need to lift its points to $\mathbb{R}^3$ to obtain the intersection curve between the considered quadrics. According to the next proposition, the line $p_1(x, y) = q_1(x, y)$ plays an important role, because those points in the cutcurve outside this line are very easy to lift.

Proposition 4.1. *If (α, β) is a point in the cutcurve such that $q_1(\alpha, \beta) \neq p_1(\alpha, \beta)$, then the z-coordinate of the point in the intersection curve is given by:*

$$z = \frac{p_0(\alpha, \beta) - q_0(\alpha, \beta)}{q_1(\alpha, \beta) - p_1(\alpha, \beta)}.$$

If (α, β) is a point of the cutcurve such that $q_1(\alpha, \beta) = p_1(\alpha, \beta)$, then the lifting of this singular point can be made by using $g(\alpha, \beta, z) = 0$ or $f(\alpha, \beta, z) = 0$.

Proof. If $(\alpha,\beta)\in\Pi(\mathcal{E}_1\cap\mathcal{E}_2)$, then there exists $z\in\mathbb{R}$ such that $f(\alpha,\beta,z)=g(\alpha,\beta,z)=0$. If $q_1(\alpha,\beta)\neq p_1(\alpha,\beta)$, then

$$g(\alpha,\beta,z)-f(\alpha,\beta,z)=(q_1(\alpha,\beta)-p_1(\alpha,\beta))z+(q_0(\alpha,\beta)-p_0(\alpha,\beta))=0$$

and

$$z=\frac{p_0(\alpha,\beta)-q_0(\alpha,\beta)}{q_1(\alpha,\beta)-p_1(\alpha,\beta)}$$

as desired. □

Previous proposition is the reason why, in this section, we will show that those points in the cutcurve and in the line $p_1(x,y)=q_1(x,y)$ are singular, and we will provide easy to use tools to compute them. However, we will also demonstrate how to compute the rest of the singular points and any other relevant points, thus enabling characterization of the cutcurve characteristics.

4.1. Characterizing the singular points of the cutcurve of $\mathcal{E}_1$ and $\mathcal{E}_2$

We will start by characterizing the singular points of the cutcurve, because they will be the most complicated points to work with when lifting the cutcurve of $\mathcal{E}_1$ and $\mathcal{E}_2$ to $\mathcal{E}_1\cap\mathcal{E}_2$. Let f and g be the polynomials in $\mathbb{R}[x,y,z]$ defining the quadrics $\mathcal{E}_1$ and $\mathcal{E}_2$:

$$f(x,y,z)=z^2+p_1(x,y)z+p_0(x,y)\qquad g(x,y,z)=z^2+q_1(x,y)z+q_0(x,y)$$

with $\deg(p_1)\leq 1$, $\deg(p_0)\leq 2$, $\deg(q_1)\leq 1$ and $\deg(q_0)\leq 2$. We restrict our attention to this case because this is the most complicated situation that we must work with: for those quadrics whose equations have a different (and simpler) structure, the singular points of the cutcurve are easier to compute, because their lifting will be given automatically by one of the two equations (being of degree in z less than or equal to 1).

The next results will be very helpful in the analysis of the singular points of the cutcurve. The first one (Theorem 4.3) shows that those points in the cutcurve and in the line $p_1(x,y)=q_1(x,y)$ are always singular points of the cutcurve (as points of the curve $\mathbf{S}_0(x_0,y_0)=0$). The next lemma provides a convenient description of the partial derivatives of $\mathbf{S}_0(x,y)$.

Lemma 4.2. *If $\mathbf{S}_0(x_0,y_0)=0$, then there exists z_0 such that $f(x_0,y_0,z_0)=0$, $g(x_0,y_0,z_0)=0$,*

$$\frac{\partial\mathbf{S}_0}{\partial x}(x_0,y_0)=\begin{vmatrix}1 & p_1(x_0,y_0) & p_0(x_0,y_0) & z_0f_x(x_0,y_0,z_0)\\ 0 & 1 & p_1(x_0,y_0) & f_x(x_0,y_0,z_0)\\ 1 & q_1(x_0,y_0) & q_0(x_0,y_0) & z_0g_x(x_0,y_0,z_0)\\ 0 & 1 & q_1(x_0,y_0) & g_x(x_0,y_0,z_0)\end{vmatrix},\ \textit{and}$$

$$\frac{\partial\mathbf{S}_0}{\partial y}(x_0,y_0)=\begin{vmatrix}1 & p_1(x_0,y_0) & p_0(x_0,y_0) & z_0f_y(x_0,y_0,z_0)\\ 0 & 1 & p_1(x_0,y_0) & f_y(x_0,y_0,z_0)\\ 1 & q_1(x_0,y_0) & q_0(x_0,y_0) & z_0g_y(x_0,y_0,z_0)\\ 0 & 1 & q_1(x_0,y_0) & g_y(x_0,y_0,z_0)\end{vmatrix}.$$

Proof. The existence of z_0 comes from the fact that $\mathbf{S}_0(x,y)$ is the resultant of $f(x,y,z)$ and $g(x,y,z)$ with respect to z. By performing elementary operations on the columns of the matrix giving $\mathbf{S}_0(x,y)$, we obtain:

$$\mathbf{S}_0(x,y)=\mathbf{Resultant}(f,g;z)=\begin{vmatrix}1 & p_1 & p_0 & 0\\ 0 & 1 & p_1 & p_0\\ 1 & q_1 & q_0 & 0\\ 0 & 1 & q_1 & q_0\end{vmatrix}=\begin{vmatrix}1 & p_1 & p_0 & zf\\ 0 & 1 & p_1 & f\\ 1 & q_1 & q_0 & zg\\ 0 & 1 & q_1 & g\end{vmatrix}.$$

Applying the rule for calculating the derivative of a determinant, we have:

$$\frac{\partial\mathbf{S}_0}{\partial x}=\begin{vmatrix}0 & p_1 & p_0 & zf\\ 0 & 1 & p_1 & f\\ 0 & q_1 & q_0 & zg\\ 0 & 1 & q_1 & g\end{vmatrix}+\begin{vmatrix}1 & p_{1x} & p_0 & zf\\ 0 & 0 & p_1 & f\\ 1 & q_{1x} & q_0 & zg\\ 0 & 0 & q_1 & g\end{vmatrix}+\begin{vmatrix}1 & p_1 & p_{0x} & zf\\ 0 & 1 & p_{1x} & f\\ 1 & q_1 & q_{0x} & zg\\ 0 & 1 & q_{1x} & g\end{vmatrix}+\begin{vmatrix}1 & p_1 & p_0 & zf_x\\ 0 & 1 & p_1 & f_x\\ 1 & q_1 & q_0 & zg_x\\ 0 & 1 & q_1 & g_x\end{vmatrix}.$$

Evaluating both sides of this equation at (x_0,y_0,z_0) produces the first equality. The second one is obtained in the same manner: it is sufficient to replace x with y when taking derivatives in the previous equation. □

Theorem 4.3. *If $\mathbf{S}_0(x_0,y_0)=0$ and $p_1(x_0,y_0)=q_1(x_0,y_0)$, then*

$$\frac{\partial\mathbf{S}_0}{\partial x}(x_0,y_0)=0\qquad\frac{\partial\mathbf{S}_0}{\partial y}(x_0,y_0)=0$$

and (x_0,y_0) is a singular point of the cutcurve.

Proof. Because $p_1(x_0, y_0) = q_1(x_0, y_0)$, by using Lemmas 4.2 and 4.4, we obtain

$$\frac{\partial \mathbf{S}_0}{\partial x}(x_0, y_0) = \begin{vmatrix} 1 & p_1(x_0, y_0) & p_0(x_0, y_0) & z_0 f_x(x_0, y_0, z_0) \\ 0 & 1 & p_1(x_0, y_0) & f_x(x_0, y_0, z_0) \\ 1 & p_1(x_0, y_0) & p_0(x_0, y_0) & z_0 g_x(x_0, y_0, z_0) \\ 0 & 1 & p_1(x_0, y_0) & g_x(x_0, y_0, z_0) \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & p_1(x_0, y_0) & p_0(x_0, y_0) & z_0 f_x(x_0, y_0, z_0) \\ 0 & 1 & p_1(x_0, y_0) & f_x(x_0, y_0, z_0) \\ 0 & 0 & 0 & z_0 (f_x(x_0, y_0, z_0) - g_x(x_0, y_0, z_0)) \\ 0 & 0 & 0 & f_x(x_0, y_0, z_0) - g_x(x_0, y_0, z_0) \end{vmatrix} = 0.$$

Replacing x with y in the previous statement, we also prove that $\frac{\partial \mathbf{S}_0}{\partial y}(x_0, y_0) = 0$. □

Below, we often use the following easily proven lemma, which is a direct consequence of the form of $\mathbf{S}_0(x, y) = 0$ presented in (3).

Lemma 4.4. *The polynomial systems of equations $\mathbf{S}_0(x, y) = 0$, $p_1(x, y) = q_1(x, y)$ and $p_1(x, y) = q_1(x, y)$, $p_0(x, y) = q_0(x, y)$ have exactly the same solutions.*

Lemma 4.4 and Theorem 4.3 imply that the singular points of the cutcurve in the line $p_1(x, y) = q_1(x, y)$ are exactly those points in the intersection between this line and the conic $p_0(x, y) = q_0(x, y)$.

Next, we study the singular points of the cutcurve outside the line $p_1(x, y) = q_1(x, y)$: they will be the projection of some special intersection points between the considered quadrics (either tangential or singular points of one of the quadrics). The proof of the next theorem is an easy consequence of the following lemma, giving a more precise description for the partial derivatives of $\mathbf{S}_0(x_0, y_0)$ when (x_0, y_0) is not in the line $p_1(x, y) = q_1(x, y)$ (and improving Lemma 4.2).

Lemma 4.5. *If (x_0, y_0) is a point of the cutcurve, $p_1(x_0, y_0) \neq q_1(x_0, y_0)$ and*

$$z_0 = -\frac{q_0(x_0, y_0) - p_0(x_0, y_0)}{q_1(x_0, y_0) - p_1(x_0, y_0)}$$

then $f(x_0, y_0, z_0) = g(x_0, y_0, z_0) = 0$ and

$$\frac{\partial \mathbf{S}_0}{\partial x}(x_0, y_0) = (q_1(x_0, y_0) - p_1(x_0, y_0)) \cdot \begin{vmatrix} f_x(x_0, y_0, z_0) & f_z(x_0, y_0, z_0) \\ g_x(x_0, y_0, z_0) & g_z(x_0, y_0, z_0) \end{vmatrix}$$

$$\frac{\partial \mathbf{S}_0}{\partial y}(x_0, y_0) = (q_1(x_0, y_0) - p_1(x_0, y_0)) \cdot \begin{vmatrix} f_y(x_0, y_0, z_0) & f_z(x_0, y_0, z_0) \\ g_y(x_0, y_0, z_0) & g_z(x_0, y_0, z_0) \end{vmatrix}.$$

Proof. We prove only the first equality: to prove the second, it is sufficient to replace x with y. Let (x_0, y_0) be such that $\mathbf{S}_0(x_0, y_0) = 0$ and z_0, such that:

$$f(x_0, y_0, z_0) = z_0^2 + p_1(x_0, y_0)z_0 + p_0(x_0, y_0) \quad g(x_0, y_0, z_0) = z_0^2 + q_1(x_0, y_0)z_0 + q_0(x_0, y_0) = 0.$$

By using Lemma 4.2, we have:

$$\frac{\partial \mathbf{S}_0}{\partial x}(x_0, y_0) = \begin{vmatrix} 1 & p_1(x_0, y_0) & p_0(x_0, y_0) & z_0 f_x(x_0, y_0, z_0) \\ 0 & 1 & p_1(x_0, y_0) & f_x(x_0, y_0, z_0) \\ 1 & q_1(x_0, y_0) & q_0(x_0, y_0) & z_0 g_x(x_0, y_0, z_0) \\ 0 & 1 & q_1(x_0, y_0) & g_x(x_0, y_0, z_0) \end{vmatrix} = \begin{vmatrix} 1 & p_1 & p_0 & z_0 f_x \\ 0 & 1 & p_1 & f_x \\ 1 & q_1 & q_0 & z_0 g_x \\ 0 & 1 & q_1 & g_x \end{vmatrix}.$$

Then

$$\frac{\partial \mathbf{S}_0}{\partial x}(x_0, y_0) = \begin{vmatrix} 1 & p_1 & p_0 & z_0 f_x \\ 0 & 1 & p_1 & f_x \\ 0 & q_1 - p_1 & q_0 - p_0 & z_0 (g_x - f_x) \\ 0 & 0 & q_1 - p_1 & g_x - f_x \end{vmatrix} = \begin{vmatrix} 1 & p_1 & f_x \\ q_1 - p_1 & q_0 - p_0 & z_0 (g_x - f_x) \\ 0 & q_1 - p_1 & g_x - f_x \end{vmatrix} =$$

$$= \begin{vmatrix} q_0 - p_0 & z_0 (g_x - f_x) \\ q_1 - p_1 & g_x - f_x \end{vmatrix} - (q_1 - p_1) \begin{vmatrix} p_1 & f_x \\ q_1 - p_1 & g_x - f_x \end{vmatrix}.$$

Because $p_1(x_0, y_0) \neq q_1(x_0, y_0)$, then

$$-z_0 (q_1(x_0, y_0) - p_1(x_0, y_0)) = q_0(x_0, y_0) - p_0(x_0, y_0)$$

and

$$\begin{aligned}\frac{\partial \mathbf{S}_0}{\partial x}(x_0,y_0) &= \begin{vmatrix} -z_0(q_1-p_1) & z_0(g_x-f_x) \\ q_1-p_1 & g_x-f_x \end{vmatrix} - (q_1-p_1)\begin{vmatrix} p_1 & f_x \\ q_1 & g_x \end{vmatrix} = \\ &= \begin{vmatrix} -z_0 & z_0 \\ 1 & 1 \end{vmatrix}(q_1-p_1)(g_x-f_x) - (q_1-p_1)\begin{vmatrix} p_1 & f_x \\ q_1 & g_x \end{vmatrix} = \\ &= (q_1-p_1)\left(-2z_0(g_x-f_x) - \begin{vmatrix} p_1 & f_x \\ q_1 & g_x \end{vmatrix}\right).\end{aligned}$$

Because $f_z(x_0,y_0,z_0) = 2z_0 + p_1(x_0,y_0)$ and $g_z(x_0,y_0,z_0) = 2z_0 + q_1(x_0,y_0)$, we have:

$$p_1 = f_z - 2z_0 \qquad q_1 = g_z - 2z_0$$

and

$$\begin{aligned}\frac{\partial \mathbf{S}_0}{\partial x}(x_0,y_0) &= (q_1-p_1)\left(-2z_0(g_x-f_x) - \begin{vmatrix} f_z-2z_0 & f_x \\ g_z-2z_0 & g_x \end{vmatrix}\right) = \\ &= (q_1-p_1)(-2z_0g_x + 2z_0f_x - g_x(f_z-2z_0) + f_x(g_z-2z_0)) = \\ &= (q_1-p_1)(f_xg_z - f_zg_x).\end{aligned}$$

Finally, we have

$$\frac{\partial \mathbf{S}_0}{\partial x}(x_0,y_0) = (q_1(x_0,y_0) - p_1(x_0,y_0)) \cdot \begin{vmatrix} f_x(x_0,y_0,z_0) & f_z(x_0,y_0,z_0) \\ g_x(x_0,y_0,z_0) & g_z(x_0,y_0,z_0) \end{vmatrix}$$

as desired □

Theorem 4.6. *Let (x_0,y_0) be a point of the cutcurve such that $p_1(x_0,y_0) \neq q_1(x_0,y_0)$ and*

$$z_0 = -\frac{q_0(x_0,y_0) - p_0(x_0,y_0)}{q_1(x_0,y_0) - p_1(x_0,y_0)}.$$

The following statements are equivalent:

1. *(x_0,y_0) is a singular point of the cutcurve.*
2. *(x_0,y_0,z_0) is a singular point of $\mathcal{E}_1$ or $\mathcal{E}_2$, or the quadrics $\mathcal{E}_1$ and $\mathcal{E}_2$ have the same tangent plane at (x_0,y_0,z_0).*

Proof. Let (x_0,y_0) be a singular point of the cutcurve such that $p_1(x_0,y_0) \neq q_1(x_0,y_0)$. Lemma 4.5 implies

$$\begin{vmatrix} f_x(x_0,y_0,z_0) & f_z(x_0,y_0,z_0) \\ g_x(x_0,y_0,z_0) & g_z(x_0,y_0,z_0) \end{vmatrix} = \begin{vmatrix} f_y(x_0,y_0,z_0) & f_z(x_0,y_0,z_0) \\ g_y(x_0,y_0,z_0) & g_z(x_0,y_0,z_0) \end{vmatrix} = 0$$

and that (x_0,y_0,z_0) is a singular point of $\mathcal{E}_1$ or $\mathcal{E}_2$, or $\mathcal{E}_1$ and $\mathcal{E}_2$ have the same tangent plane at (x_0,y_0,z_0).

If (x_0,y_0,z_0) is a singular point of $\mathcal{E}_1$ or $\mathcal{E}_2$ such that $p_1(x_0,y_0) \neq q_1(x_0,y_0)$, then the equalities in Lemma 4.5 imply that (x_0,y_0) is a singular point of the cutcurve. If (x_0,y_0,z_0) is a tangential intersection point of $\mathcal{E}_1$ or $\mathcal{E}_2$ such that $p_1(x_0,y_0) \neq q_1(x_0,y_0)$, then

$$\begin{vmatrix} f_x(x_0,y_0,z_0) & f_z(x_0,y_0,z_0) \\ g_x(x_0,y_0,z_0) & g_z(x_0,y_0,z_0) \end{vmatrix} = \begin{vmatrix} f_y(x_0,y_0,z_0) & f_z(x_0,y_0,z_0) \\ g_y(x_0,y_0,z_0) & g_z(x_0,y_0,z_0) \end{vmatrix} = \begin{vmatrix} f_x(x_0,y_0,z_0) & f_y(x_0,y_0,z_0) \\ g_x(x_0,y_0,z_0) & g_y(x_0,y_0,z_0) \end{vmatrix} = 0$$

and the equalities in Lemma 4.5 imply that (x_0,y_0) is a singular point of the cutcurve. □

The next corollary summarizes the results obtained regarding the singular points of the cutcurve and how they can be classified (according to the way they will be computed).

Corollary 4.7. *The singular points of the cutcurve come from three different sources:*

1. *they are points of the cutcurve in the line $p_1(x,y) = q_1(x,y)$ (exactly those in this line and in the conic $p_0(x,y) = q_0(x,y)$), or*
2. *they are points of the cutcurve not in the line $p_1(x,y) = q_1(x,y)$ coming from the projection of a tangential intersection point between $\mathcal{E}_1$ and $\mathcal{E}_2$, or*
3. *they are points of the cutcurve not in the line $p_1(x,y) = q_1(x,y)$ coming from the projection of an intersection point between $\mathcal{E}_1$ and $\mathcal{E}_2$, which is a singular point of $\mathcal{E}_1$ or $\mathcal{E}_2$.*

Proof. Theorem 4.3 and Lemma 4.4 imply (1). Theorem 4.6 implies (2) and (3). □

Singular points of the cutcurve in the line $p_1(x,y) = q_1(x,y)$ will be computed by intersecting this line with the conic $p_0(x,y) = q_0(x,y)$ (see Section 4.3). Singular points of the cutcurve outside the line $p_1(x,y) = q_1(x,y)$ will be computed

in Section 4.4. To decide where the singular points of the cutcurve outside this line come from, it is sufficient to take into account that the singular points of a quadric can be easily computed (when they exist, by just solving a linear system of equations and determining whether the solutions belong to the quadric). Therefore, we will always know exactly the source (according to Corollary 4.7) for every singular point of the cutcurve.

Furthermore, not all tangential intersection points between $\mathcal{E}_1$ and $\mathcal{E}_2$ will come from the second option in the previous corollary: the projection of such a point may lie on the line $p_1(x, y) = q_1(x, y)$ (see Example 5.2 for a particular case of this situation).

4.2. Characterizing the critical points of the cutcurve (and computing the regular points in the critical lines)

To determine the critical points of the cutcurve, we must solve the system of equations

$$\mathbf{S}_0(x, y) = 0, \quad \frac{\partial \mathbf{S}_0}{\partial y}(x, y) = 0 \tag{5}$$

inside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$. We will solve this system by using subresultants and the possible factorizations of a degree 4 univariate polynomial.

Because $\mathbf{S}_0(x, y)$ has total degree bounded by 4, let us assume first that $\deg(\mathbf{S}_0, y) = 4$

$$\mathbf{S}_0(x, y) = \tau_4 y^4 + \tau_3(x)y^3 + \tau_2(x)y^2 + \tau_1(x)y + \tau_0(x)$$

with $\tau_4 \in \mathbb{R} - \{0\}$, $\tau_i(x) \in \mathbb{R}[x]$ and $\deg(\tau_i(x)) \leq 4 - i$ $(0 \leq i \leq 3)$, together with $\mathbf{S}_0(x, y)$ squarefree. We compute the following polynomials

$$L_2(x, y) = \mathbf{Sres}_2\left(\mathbf{S}_0(x, y), \frac{\partial \mathbf{S}_0}{\partial y}(x, y); y\right) = \sigma_2(x)y^2 + \sigma_{2,1}(x)y + \sigma_{2,0}(x),$$
$$L_1(x, y) = \mathbf{Sres}_1\left(\mathbf{S}_0(x, y), \frac{\partial \mathbf{S}_0}{\partial y}(x, y); y\right) = \sigma_1(x)y + \sigma_{1,0}(x),$$
$$L_0(x, y) = \mathbf{Sres}_0\left(\mathbf{S}_0(x, y), \frac{\partial \mathbf{S}_0}{\partial y}(x, y); y\right) = \mathbf{Resultant}\left(\mathbf{S}_0(x, y), \frac{\partial \mathbf{S}_0}{\partial y}(x, y); y\right) = \sigma_0(x).$$

According to (1), if $(\alpha, \beta) \in \mathbb{R}^2$ is a solution of (5), then $\sigma_0(\alpha) = 0$ and β is a multiple root of $\mathbf{S}_0(\alpha, y)$. Thus, $\mathbf{S}_0(\alpha, y) = \tau y^4 + \cdots$ can factor only in the following ways:

a. $\mathbf{S}_0(\alpha, y) = \tau_4(y - \beta)^2(y - \gamma_1)(y - \gamma_2)$ with $\gamma_1 \neq \gamma_2$ (if $\gamma_i \in \mathbb{C} - \mathbb{R}$, then $\gamma_1 = \overline{\gamma_2}$).
b. $\mathbf{S}_0(\alpha, y) = \tau_4(y - \beta)^2(y - \gamma)^2$ with $\gamma \in \mathbb{R}$.
c. $\mathbf{S}_0(\alpha, y) = \tau_4(y - \beta)^3(y - \gamma)$ with $\gamma \in \mathbb{R}$.
d. $\mathbf{S}_0(\alpha, y) = \tau_4(y - \beta)^4$.

Polynomials $\sigma_i(x)$ and $\sigma_{i,j}(x)$ characterize each possibility in the following way (according to (2)):

A. If $\sigma_0(\alpha) = 0$ and $\sigma_1(\alpha) \neq 0$, then

$$\gcd\left(\mathbf{S}_0(x, y), \frac{\partial \mathbf{S}_0}{\partial y}(x, y)\right) = L_1(x, y), \quad \mathbf{S}_0(\alpha, y) = \tau_4(y - \beta)^2(y - \gamma_1)(y - \gamma_2),$$
$$\beta = -\frac{\sigma_{1,0}(\alpha)}{\sigma_1(\alpha)} \quad \text{and} \quad (y - \gamma_1)(y - \gamma_2) = \frac{\mathbf{S}_0(\alpha, y)}{\tau_4(y - \beta)^2}.$$

B. If $\sigma_0(\alpha) = 0$, $\sigma_1(\alpha) = 0$, $\sigma_2(\alpha) \neq 0$ and $\sigma_{2,1}(\alpha)^2 - 4\sigma_2(\alpha)\sigma_{2,0}(\alpha) > 0$, then

$$\gcd\left(\mathbf{S}_0(x, y), \frac{\partial \mathbf{S}_0}{\partial y}(x, y)\right) = L_2(x, y), \quad \mathbf{S}_0(\alpha, y) = \tau_4(y - \beta_1)^2(y - \beta_2)^2,$$
$$\sigma_2(\alpha)(y - \beta_1)(y - \beta_2) = \sigma_2(\alpha)y^2 + \sigma_{2,1}(\alpha)y + \sigma_{2,0}(\alpha) \quad \text{and}$$
$$\beta_1, \beta_2 = \frac{-\sigma_{2,1}(\alpha) \pm \sqrt{\sigma_{2,1}(\alpha)^2 - 4\sigma_2(\alpha)\sigma_{2,1}(\alpha)}}{2\sigma_2(\alpha)}.$$

C. If $\sigma_0(\alpha) = 0$, $\sigma_1(\alpha) = 0$, $\sigma_2(\alpha) \neq 0$ and $\sigma_{2,1}(\alpha)^2 - 4\sigma_2(\alpha)\sigma_{2,0}(\alpha) = 0$, then

$$\gcd\left(\mathbf{S}_0(x, y), \frac{\partial \mathbf{S}_0}{\partial y}(x, y)\right) = L_2(x, y), \quad \mathbf{S}_0(\alpha, y) = \tau_4(y - \beta)^3(y - \gamma),$$
$$\beta = -\frac{\sigma_{2,1}(\alpha)}{2\sigma_2(\alpha)}, \quad y - \gamma = \frac{\mathbf{S}_0(\alpha, y)}{\tau_4(y - \beta)^3} \quad \text{and} \quad \gamma = \frac{\tau_0(\alpha)}{\tau_4\beta^3}.$$

D. If $\sigma_0(\alpha) = 0$, $\sigma_1(\alpha) = 0$ and $\sigma_2(\alpha) = 0$, then

$$\gcd\left(\mathbf{S}_0(\alpha, y), \frac{\partial \mathbf{S}_0}{\partial y}(\alpha, y)\right) = \frac{\partial \mathbf{S}_0}{\partial y}(\alpha, y) = 4\tau_4 y^3 + 3\tau_3(\alpha)y^2 + 2\tau_2(\alpha)y + \tau_1(\alpha),$$

$$\mathbf{S}_0(\alpha, y) = \tau_4(y - \beta)^4 \quad \text{and} \quad \beta = -\frac{\tau_3(\alpha)}{4\tau_4}.$$

For any $(\alpha, \beta) \in \mathbb{R}^2$ solution of the system (5), α is a real root of $\sigma_0(x)$, and there is a function $\Phi(x)$ such that $\beta = \Phi(\alpha)$. In this way, we have characterized the critical points of the cutcurve: it is sufficient to keep those points verifying $(\alpha, \beta) \in \mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$.

Because we will use this result to compute the critical points of the cutcurve after the real roots of $\sigma_0(x)$ have been determined, there is one more case to take into account when α is a real root of $\sigma_0(x)$:

e. $\mathbf{S}_0(\alpha, y) = \tau_4(y - \gamma)^2(y - \overline{\gamma})^2$ with $\gamma \in \mathbb{C} - \mathbb{R}$.

The characterization of this case is provided by:

E. $\sigma_0(\alpha) = 0$, $\sigma_1(\alpha) = 0$, $\sigma_2(\alpha) \neq 0$ and $\sigma_{2,1}(\alpha)^2 - 4\sigma_2(\alpha)\sigma_{2,0}(\alpha) < 0$.

There is no need in this case to explicitly compute γ, as in the other cases, because γ is complex and non real number.

Previous discussion also provides the method for computing the regular points (the γ's) in each critical line $x = \alpha$ when they exist: by a direct formula or solving a quadratic equation.

Cases in which $\deg_y(\mathbf{S}_0) \leq 3$ are treated in a similar way, and they are easier to work with than those analyzed before in which $\deg_y(\mathbf{S}_0) = 4$. Again, the subresultants of $\mathbf{S}_0(x, y)$ and its partial derivative with respect to y are used. In this case, special attention must be paid to the leading coefficient of $\mathbf{S}_0(x, y)$ with respect to y, because it will be appear as a factor of $\sigma_0(x)$ whose real roots will produce vertical asymptotes of the cutcurve (always considered these lines inside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$).

If $\mathbf{S}_0(x, y)$ is not squarefree, then the problem of analyzing the cutcurve is easier to work with. This situation occurs when $\sigma_0(x)$ is identically zero, and, removing the multiple factor from $\mathbf{S}_0(x, y)$ (through gcd computations), we conclude that the cutcurve is a conic and a line, a conic, two different lines or one line (always inside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$). In this case, for completeness, we will use $\sigma_0(x)$ to denote the resultant of the squarefree part of $\mathbf{S}_0(x, y)$ and its partial derivative with respect to y.

4.3. Computing the singular points of the cutcurve in the line $p_1(x, y) = q_1(x, y)$

Theorem 4.3 and Lemma 4.4 imply that the singular points of the cutcurve in the line $p_1(x, y) = q_1(x, y)$ are exactly those points in this line also in the conic $p_0(x, y) = q_0(x, y)$ that can be computed by just solving a degree two equation. In some cases, the line $p_1(x, y) - q_1(x, y) = 0$ (or a part of it) is contained completely in the cutcurve: this possibility can be directly verified by performing the corresponding substitution in $\mathbf{S}_0(x, y)$.

In the particular case when $p_1(x, y) \equiv q_1(x, y)$, we have

$$\mathbf{S}_0(x, y) = (p_0(x, y) - q_0(x, y))^2$$

and all points in the cutcurve are singular. However, this implies that the cutcurve is contained in the conic $p_0(x, y) = q_0(x, y)$, and all further computations (including the lifting) are greatly simplified, because we can explicitly describe y in terms of x (involving radicals) or even rationally parameterize the cutcurve, because it is a conic (or a part of it).

When $p_0(x, y) \equiv q_0(x, y)$, we have

$$\mathbf{S}_0(x, y) = p_0(x, y)(p_1(x, y) - q_1(x, y))^2 = q_0(x, y)(p_1(x, y) - q_1(x, y))^2$$

and all points in the line $p_1(x, y) = q_1(x, y)$ belonging to $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$ are in the cutcurve (and are singular). In this case, the analysis of the cutcurve is very easy, because it requires the analysis of the line $p_1(x, y) = q_1(x, y)$ and the conic $p_0(x, y) = 0$ inside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$. The lifting is also easy, through the same strategy as before.

When $p_0(x, y) \not\equiv q_0(x, y)$, $p_1(x, y) \not\equiv q_1(x, y)$ and $\deg(p_1(x, y) - q_1(x, y)) = 0$, there are no singular points of the cutcurve in the line $p_1(x, y) = q_1(x, y)$.

In the general case, $p_0(x, y) \not\equiv q_0(x, y)$, $p_1(x, y) \not\equiv q_1(x, y)$ and $\deg(p_1(x, y) - q_1(x, y)) = 1$, the singular points of the cutcurve in the line $p_1(x, y) - q_1(x, y)$ are determined by intersecting this line with the conic $p_0(x, y) - q_0(x, y)$.

4.4. Computing the singular points of the cutcurve not in the line $p_1(x, y) = q_1(x, y)$

We show here how to determine the singular points of the cutcurve not belonging to the line $p_1(x, y) = q_1(x, y)$. These points, according to Theorem 4.6, come from the projection of a tangential intersection point of the two considered quadrics or from an intersection point which is a singular point of one of them. They are very easily lifted, because they are not in the line $p_1(x, y) = q_1(x, y)$. To determine these points, we must solve the system of equations

$$\mathbf{S}_0(x, y) = 0, \quad \frac{\partial \mathbf{S}_0}{\partial x}(x, y) = 0 \quad \frac{\partial \mathbf{S}_0}{\partial y}(x, y) = 0, \quad p_1(x, y) \neq q_1(x, y) \tag{6}$$

inside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$. Because we have already characterized the critical points of the cutcurve, to determine the singular points of the cutcurve not belonging to the line $p_1(x,y) = q_1(x,y)$, we must only check, for every candidate $(\alpha, \Phi(\alpha))$ (with $\alpha \in \mathbb{R}$ such that $\sigma_0(\alpha) = 0$), that

$$\frac{\partial \mathbf{S}_0}{\partial x}(\alpha, \Phi(\alpha)) = 0 \quad \text{and} \quad p_1(\alpha, \Phi(\alpha)) \neq q_1(\alpha, \Phi(\alpha)).$$

Some univariate gcd computations are needed to determine the factors of σ_0 producing the singular points of the cutcurve outside the line $p_1(x,y) = q_1(x,y)$. For example, when α comes from case (c) in Section 4.2, those values of α producing the searched singular points of the cutcurve are the real roots of the polynomial

$$T_3(x) = \frac{S_3(x)}{\gcd\left(S_3(x), \sigma_2(x), p_1(x, \Phi(x)) - q_1(x, \Phi(x))\right)}$$

where

$$S_3(x) = \gcd\left(\sigma_0(x), \sigma_1(x), \sigma_{2,1}(x)^2 - 4\sigma_2(x)\sigma_{2,0}(x), \text{numerator}\left(\frac{\partial \mathbf{S}_0}{\partial x}(x, \Phi(x))\right)\right).$$

If α comes from case (b) in Section 4.2, those values of α producing the searched singular points of the cutcurve are the real roots of the polynomial

$$T_2(x) = \frac{S_2(x)}{\gcd\left(S_2(x), \sigma_2(x), p_1(x, \Phi(x)) - q_1(x, \Phi(x))\right)}$$

such that $\sigma_{2,1}(x)^2 - 4\sigma_2(x)\sigma_{2,0}(x) > 0$, where

$$S_2(x) = \gcd\left(\sigma_0(x), \sigma_1(x), \textbf{Resultant}\left(\frac{\partial \mathbf{S}_0}{\partial x}(x,y), \sigma_2(x)y^2 + \sigma_{2,1}(x)y + \sigma_{2,0}(x); y\right)\right).$$

Formulae when α comes from cases (a) or (d) are similar to those shown before for case (c). After the polynomials $T_i(\alpha)$ have been determined, their real roots can be approximated numerically to produce the searched singular points of the cutcurve and the corresponding intersection points between the considered quadrics.

4.5. The factorization of $\sigma_0(x)$

Previous subsections have shown that we need to work with the real roots of the factors of $\sigma_0(x)$. Therefore, we analyze the degree of $\sigma_0(x)$ and how it factors, because, in many cases, we already know some of its roots (those coming from the singular points of the cutcurve belonging to the line $p_1(x,y) = q_1(x,y)$). The main conclusion from this section is that degree of any factor of $\sigma_0(x)$ is less than or equal to 8 (when its degree might be equal to 12). This implies, for example, that we will never need to compute the real roots of a polynomial whose degree is greater than 8.

The analysis of $\sigma_0(x)$ will be performed by considering two generic polynomials $f(x,y,z)$ and $g(x,y,z)$

$$f(x,y,z) = z^2 + (a_{101}x + a_{011}y + a_{001})z + a_{200}x^2 + a_{110}xy + a_{020}y^2 + a_{100}x + a_{010}y + a_{000}$$
$$g(x,y,z) = z^2 + (b_{101}x + b_{011}y + b_{001})z + b_{200}x^2 + b_{110}xy + b_{020}y^2 + b_{100}x + b_{010}y + b_{000}$$

and using `Maple` to symbolically compute both $\mathbf{S}_0(x,y)$ and $\sigma_0(x)$, taking into account the different possibilities for the degrees of $p_1(x,y) - q_1(x,y)$ and $p_0(x,y) - q_0(x,y)$. The next theorem shows the main conclusion from the analysis of $\sigma_0(x)$ performed with `Maple`.

Theorem 4.8. *The degree of any factor of $\sigma_0(x)$ is less than or equal to 8.*

Proof. The degree of $\sigma_0(x)$ is bounded by 12 (being the discriminant with respect to y of $\mathbf{S}_0(x,y)$ whose total degree is bounded by 4).

The cases $p_1(x,y) \equiv q_1(x,y)$ or $p_0(x,y) \equiv q_0(x,y)$ have already been analyzed in Section 4.3, and they are very easy to work with, because the whole cutcurve is, respectively, a conic, $p_0(x,y) - q_0(x,y)$, or the union of a line and a conic, $(p_0(x,y) - q_0(x,y))(p_1(x,y) - q_1(x,y))$ (inside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$).

When $p_0(x,y) \not\equiv q_0(x,y)$, $p_1(x,y) \not\equiv q_1(x,y)$ and $\deg(p_1(x,y) - q_1(x,y)) = 0$, there are no singular points of the cutcurve in the line $p_1(x,y) = q_1(x,y)$. In this case, the degree of $\sigma_0(x)$ is bounded by 8. In the particular case $a_{020} = b_{020}$, the degree of $\sigma_0(x)$ is bounded by 6 and factors, as the product of a degree 2 polynomial times a degree 4 polynomial. In this last case, the degree 2 polynomial starts

$$(a_{110} - b_{110})^2 x^2 - (a_{110} - b_{110})(a_{001}a_{011} - a_{011}b_{001} - 2a_{010} + 2b_{010})x + \cdots.$$

When $p_0(x,y) \not\equiv q_0(x,y)$, $p_1(x,y) \not\equiv q_1(x,y)$, $\deg_x(p_1(x,y) - q_1(x,y)) = 1$ and $\deg_y(p_1(x,y) - q_1(x,y)) = 0$, the linear factor

$$W(x) = p_1(x,y) - q_1(x,y) = (a_{101} - b_{101})x + a_{001} - b_{001}$$

appears in $\sigma_0(x)$ in all possible cases (even if there are no intersections between $p_1(x,y) = q_1(x,y)$ and $p_0(x,y) = q_0(x,y)$):

- If $p_0(x,y)-q_0(x,y)$ is a non-zero constant, then $\deg_x(\sigma_0(x)) \leq 8$ and $W(x)$ is a multiplicity 4 factor of $\sigma_0(x)$ when $a_{020}=b_{020}\neq 0$. If $a_{020}=b_{020}=0$, then $\deg_x(\sigma_0(x)) \leq 3$, and the multiplicity of the linear factor $W(x)$ of $\sigma_0(x)$ is 1.
- If $\deg(p_0(x,y)-q_0(x,y))=1$, then $\deg_x(\sigma_0(x)) \leq 8$ and $W(x)$ is a multiplicity 2 factor of $\sigma_0(x)$. This polynomial has two more factors: one of a degree less than or equal to 4 and one of a degree less than or equal to 2:

$$a_{020}\,(a_{101}-b_{101})^2\,x^2+(a_{101}-b_{101})\,(2a_{001}a_{020}-a_{010}a_{011}+a_{011}b_{010}-2a_{020}b_{001})\,x+\cdots$$

- If $\deg(p_0(x,y)-q_0(x,y))=2$ and $b_{020}=a_{020}$, then $\deg_x(\sigma_0(x)) \leq 8$ and $W(x)$ is a multiplicity 2 factor of $\sigma_0(x)$. This polynomial has two more factors: one of degree less than or equal to 4 and one of degree less than or equal to 2.
- If $\deg(p_0(x,y)-q_0(x,y))=2$ and $b_{020}\neq a_{020}$, then $\deg_x(\sigma_0(x)) \leq 12$ and $W(x)$ is a multiplicity 4 factor of $\sigma_0(x)$.

When $p_0(x,y)\not\equiv q_0(x,y)$, $p_1(x,y)\not\equiv q_1(x,y)$ and $\deg_y(p_1(x,y)-q_1(x,y))=1$, we define:

$$\ell(x)=-\frac{(a_{101}-b_{101})x+a_{001}-b_{001}}{a_{011}-b_{011}},\quad U(x)=p_0(x,\ell(x))-q_0(x,\ell(x)).$$

In this case, the properties of the $\sigma_0(x)$ factors are the following:

- If $p_0(x,y)-q_0(x,y)$ is a non-zero constant and $b_{020}=a_{020}=0$, then $\deg_x(\sigma_0(x)) \leq 7$ and $a_{110}x+a_{010}$ is a multiplicity 1 factor of $\sigma_0(x)$.
- If $p_0(x,y)-q_0(x,y)$ is a non-zero constant and $b_{020}=a_{020}\neq 0$, then $\deg_x(\sigma_0(x)) \leq 8$.
- If $\deg(p_0(x,y)-q_0(x,y))=1$ and $b_{020}=a_{020}=0$, then $\deg_x(\sigma_0(x)) \leq 9$ and the linear factor $U(x)$ is a multiplicity 2 factor of $\sigma_0(x)$. This polynomial has one more linear factor:

$$(a_{011}a_{110}-a_{110}b_{011})x-a_{010}b_{011}+a_{011}b_{010}$$

- If $\deg(p_0(x,y)-q_0(x,y))=1$ and $b_{020}=a_{020}\neq 0$, then $\deg_x(\sigma_0(x)) \leq 10$, and the linear factor $U(x)$ is a multiplicity 2 factor of $\sigma_0(x)$.
- If $\deg(p_0(x,y)-q_0(x,y))=2$, then $\deg_x(\sigma_0(x)) \leq 12$ and the quadratic factor $U(x)$ is a multiplicity 2 factor of $\sigma_0(x)$.

Therefore, the unique cases in which $\deg_x(\sigma_0(x))=12$ are

- when $\deg_x(p_1(x,y)-q_1(x,y))=1$, $\deg_y(p_1(x,y)-q_1(x,y))=0$, $\deg(p_0(x,y)-q_0(x,y))=2$ and $b_{020}\neq a_{020}$ and when the linear factor $W(x)$ is a multiplicity 4 factor of $\sigma_0(x)$, and
- when $\deg_y(p_1(x,y)-q_1(x,y))=1$ and $\deg(p_0(x,y)-q_0(x,y))=2$ and when the quadratic factor $U(x)$ is a multiplicity 2 factor of $\sigma_0(x)$.

In these two cases, $\sigma_0(x)$ has a degree 4 factor that we know explicitly. The remaining three cases in which we also face a degree 8 factor of $\sigma_0(x)$ are the following:

- When $p_1(x,y)\not\equiv q_1(x,y)$, $\deg(p_1(x,y)-q_1(x,y))=0$ and $a_{020}\neq b_{020}$ when $\sigma_0(x)$ has degree 8.
- When $\deg_y(p_1(x,y)-q_1(x,y))=1$, $p_0(x,y)-q_0(x,y)$ is a non-zero constant and $b_{020}=a_{020}\neq 0$ when $\sigma_0(x)$ has degree 8.
- When $\deg_y(p_1(x,y)-q_1(x,y))=1$, $\deg(p_0(x,y)-q_0(x,y))=1$ and $b_{020}=a_{020}\neq 0$ when $\sigma_0(x)$ has degree 10 and the linear factor $U(x)$ is a multiplicity 2 factor of $\sigma_0(x)$.

In the most generic case, (f and g squarefree, $\deg_y(\mathbf{S}_0(x,y))=4$, $\deg(p_0(x,y)-q_0(x,y))=2$, $\deg_y(p_1(x,y)-q_1(x,y))=1$ and $U(x)$ with two different roots), it is also possible to prove that the quadratic polynomial $U(x)$ is a multiplicity two factor of $\sigma_0(x)$. This is achieved as follows:

1. Because

$$\sigma_0(x)=\mathbf{Resultant}\left(\mathbf{S}_0(x,y),\ \frac{\partial \mathbf{S}_0}{\partial y}(x,y);y\right)$$

there exist polynomials $A(x,y)$ and $B(x,y)$ such that

$$\sigma_0(x)=A(x,y)\mathbf{S}_0(x,y)+B(x,y)\frac{\partial \mathbf{S}_0}{\partial y}(x,y).$$

2. The discussion in Section 4.3 (or Theorem 4.3 and Lemma 4.4) implies that $U(x)$ divides both $\mathbf{S}_0(x,\ell(x))$ and $\frac{\partial \mathbf{S}_0}{\partial y}(x,\ell(x))$ (and $\frac{\partial \mathbf{S}_0}{\partial x}(x,\ell(x))$), and thus $U(x)$ divides $\sigma_0(x)$.
3. Let α_1 and α_2 be the two different roots of $U(x)$. Lemma 4.1 in [35] says that, given (δ,ϵ) an intersection point of two polynomials $s(x,y)$ and $r(x,y)$, δ is a multiple root of the resultant of s and r if and only if

$$\left(\frac{\partial s}{\partial x}(\delta,\epsilon)\frac{\partial r}{\partial y}(\delta,\epsilon)-\frac{\partial s}{\partial y}(\delta,\epsilon)\frac{\partial r}{\partial x}(\delta,\epsilon)\right)\mathbf{sres}_1(s,r;y)(\delta)$$

vanishes. In our case, each α_i comes from a singular point of $\mathbf{S}_0(x,y)$ and, according to the lemma from [35] (applied to $\mathbf{S}_0(x,y)$ and its partial derivative with respect to y), is a multiple root of the resultant concluding that $U(x)^2$ divides $\sigma_0(x)$.

We can conclude that, in all possible cases, we are able to explicitly decompose $\sigma_0(x)$ such that the degree of any factor is less than or equal to 8. □

If f or g are not squarefree, then one of them is the square of a linear form, and $\mathbf{S}_0(x, y)$ is the square of a degree 2 polynomial in x and y whose discriminant is $\sigma_0(x)$ (a quadratic polynomial). If f and g are not squarefree, then $\mathbf{S}_0(x, y)$ is the fourth power of a linear polynomial in x and y whose discriminant is a non-zero constant. If f or g are reducible (i.e., a product of two planes), then the cutcurve is a product of two conics and $\sigma_0(x)$ factors as the product of two degree 6 polynomials.

Remark 4.9. The proof of Theorem 4.8 contains much more detail than necessary, because it contains the explicit description of some of the factors of $\sigma_0(x)$ depending on the degrees of $p_1(x, y) - q_1(x, y)$ and $p_0(x, y) - q_0(x, y)$.

4.6. Determining the intersection points of the cutcurve with the silhouette curves

Next lemma connects the polynomials $\mathbf{S}_0(x, y)$, $p_1(x, y) - q_1(x, y)$, $\Delta_{\varepsilon_1}(x, y)$ and $\Delta_{\varepsilon_2}(x, y)$. It will be used to analyze the intersection of the cutcurve with the silhouette curves.

Lemma 4.10.

$$\mathbf{S}_0(x, y) = \frac{1}{16}\Big[(p_1 - q_1)^4 + \left(\Delta_{\varepsilon_1} - \Delta_{\varepsilon_2}\right)^2 - 2\,(p_1 - q_1)^2\left(\Delta_{\varepsilon_1} + \Delta_{\varepsilon_2}\right)\Big].$$

Proof. Because $\Delta_{\varepsilon_1} = p_1^2 - 4p_0$ and $\Delta_{\varepsilon_2} = q_1^2 - 4q_0$, we have

$$\begin{aligned}
&\frac{1}{16}\Big[(p_1 - q_1)^4 + \left(\Delta_{\varepsilon_1} - \Delta_{\varepsilon_2}\right)^2 - 2\,(p_1 - q_1)^2\left(\Delta_{\varepsilon_1} + \Delta_{\varepsilon_2}\right)\Big] = \\
&= {p_0}^2 - 2p_0q_0 + {q_0}^2 + {p_1}^2q_0 + {q_1}^2p_0 - p_1q_0q_1 - p_0p_1q_1 = \\
&= (p_0 - q_0)^2 - (p_1 - q_1)\,(p_0q_1 - q_0p_1) = \mathbf{S}_0(x, y)
\end{aligned}$$

as desired (see (3)). □

The first consequence of the previous lemma provides a method for easily determining the points in the cutcurve belonging to each silhouette curve.

Proposition 4.11. *For $i \in \{1, 2\}$, the polynomial systems of equations*

$$\mathbf{S}_0(x, y) = 0, \quad \Delta_{\varepsilon_i}(x, y) = 0$$

and

$$p_1(x, y)q_1(x, y) = 2\,(q_0(x, y) + p_0(x, y))\,, \quad \Delta_{\varepsilon_i}(x, y) = 0$$

have exactly the same solutions.

Proof. We prove only the case $i = 1$. Using Lemma 4.10, if $\mathbf{S}_0(\alpha, \beta) = 0$ and $\Delta_{\varepsilon_1}(\alpha, \beta) = 0$, then we have

$$(p_1 - q_1)^4 + \Delta_{\varepsilon_2}^2 - 2\,(p_1 - q_1)^2\,\Delta_{\varepsilon_2} = \left((p_1 - q_1)^2 - \Delta_{\varepsilon_2}\right)^2 = 0$$

and $\Delta_{\varepsilon_2} = (p_1 - q_1)^2$. Because $\Delta_{\varepsilon_1} = {p_1}^2 - 4p_0 = 0$ and $\Delta_{\varepsilon_2} = {q_1}^2 - 4q_0$, we have

$${q_1}^2 - 4q_0 = \Delta_{\varepsilon_2} = (p_1 - q_1)^2 = p_1^2 - 2p_1q_1 + q_1^2 = 4p_0 - 2p_1q_1 + q_1^2$$

and we conclude $p_1(\alpha, \beta)q_1(\alpha, \beta) = 2(p_0(\alpha, \beta) + q_0(\alpha, \beta))$.

If $p_1(\alpha, \beta)q_1(\alpha, \beta) = 2(p_0(\alpha, \beta) + q_0(\alpha, \beta))$ and $\Delta_{\varepsilon_1}(\alpha, \beta) = 0$, using Lemma 4.10, then we have

$$\begin{aligned}
\mathbf{S}_0(\alpha, \beta) &= \frac{1}{16}\Big[(p_1 - q_1)^4 + \left(\Delta_{\varepsilon_1} - \Delta_{\varepsilon_2}\right)^2 - 2\,(p_1 - q_1)^2\left(\Delta_{\varepsilon_1} + \Delta_{\varepsilon_2}\right)\Big] = \\
&= \frac{1}{16}\Big[(p_1 - q_1)^4 + \Delta_{\varepsilon_2}^2 - 2\,(p_1 - q_1)^2\,\Delta_{\varepsilon_2}\Big] = \frac{1}{16}\Big[(p_1 - q_1)^2 - \Delta_{\varepsilon_2}\Big]^2 = \\
&= \frac{1}{16}\Big[p_1^2 - 2p_1q_1 + 4q_0\Big]^2 = \frac{1}{16}\Big[p_1^2 - 4(p_0 + q_0) + 4q_0\Big]^2 = \frac{1}{16}\Big[p_1^2 - 4p_0\Big]^2 = \frac{1}{16}\Delta_{\varepsilon_1}^2 = 0
\end{aligned}$$

as desired. □

As a consequence, for solving each system

$$\mathbf{S}_0(x, y) = 0, \quad \Delta_{\varepsilon_i}(x, y) = 0$$

we solve the simpler system (intersecting two conics)

$$p_1(x,y)q_1(x,y) = 2(p_0(x,y) + q_0(x,y)), \quad \Delta_{\mathcal{E}_i}(x,y) = 0.$$

Final consequence of Lemma 4.10 shows that the vertices of the region $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$ belonging to the cutcurve are always singular points of the cutcurve. In other words, the cutcurve and both silhouette curves have no common points outside the line $p_1(x,y) = q_1(x,y)$.

Corollary 4.12. *If (α,β) is a point of the cutcurve such that $\Delta_{\mathcal{E}_1}(\alpha,\beta) = \Delta_{\mathcal{E}_2}(\alpha,\beta) = 0$, then $p_1(\alpha,\beta) = q_1(\alpha,\beta)$ and (α,β) is a singular point of the cutcurve.*

Proof. From Lemma 4.10 we have, for (α,β) a point of the cutcurve,

$$\mathbf{S}_0(\alpha,\beta) = 0,\ \Delta_{\mathcal{E}_1}(\alpha,\beta) = 0,\ \Delta_{\mathcal{E}_2}(\alpha,\beta) = 0 \Longrightarrow \left(\frac{(p_1-q_1)^2}{4}\right)^2 = 0 \Longrightarrow p_1(\alpha,\beta) = q_1(\alpha,\beta)$$

as desired. The second claim is a direct consequence of Theorem 4.3. □

4.7. Cutcurve branches and their connections

The number of branches of the curve $\mathbf{S}_0(x,y) = 0$ between two consecutive critical lines will be 0, 2 or 4: this will be determined by picking one point τ in the interval between two consecutive critical lines and determining the number of real roots of $\mathbf{S}_0(\tau,y) = 0$: in fact, the evaluation of the subresultants $\sigma_j(\tau)$, already computed, automatically provides this number; see [33,34]. Connecting the points in the critical lines and the branches is also quite simple, because:

- for the regular points in the critical lines, there will be only one branch to the left and one branch to the right,
- for the critical lines with only one critical point, once used, the branches go to the regular points, and those remaining go the critical point, and
- for the critical lines with two critical points (no regular points in this case), if four branches are shared (to the left or to the right), then two will go to each critical point, and if two branches are shared, then they will both go to one of the points.

The only remaining question to answer is, in the last possibility mentioned above (two critical points in the critical line and only two branches to share to the left or to the right), how to decide the critical point where the two branches arrive or leave. Let $x = \alpha$ be the critical line, (α,β_1) and (α,β_2) the two critical points in $x = \alpha$, and assume that the two branches that connect with these two points are to the left. Under these hypotheses, $\frac{\partial \mathbf{S}_0}{\partial y}(x,y) = 0$ is a cubic curve such that

$$\frac{\partial \mathbf{S}_0}{\partial y}(\alpha,\beta_1) = \frac{\partial \mathbf{S}_0}{\partial y}(\alpha,\beta_2) = 0.$$

By using the formulae in Section 4.2 (case 2), we determine that the third real root of $\frac{\partial \mathbf{S}_0}{\partial y}(\alpha,y)$ is

$$\gamma = \frac{\beta_1+\beta_2}{2} = -\frac{\sigma_{2,1}(\alpha)}{2\sigma_2(\alpha)}.$$

If $\kappa < \alpha$ is close enough to α, then the position of the two real roots $\gamma_1 < \gamma_2$ of $\mathbf{S}_0(\kappa,y) = 0$ with respect to γ will tell us to which critical point to connect the two branches to the left (see Fig. 2):

- If $\gamma_1 < \gamma_2 < \gamma$, then the two branches will arrive at the smallest critical point.
- If $\gamma < \gamma_1 < \gamma_2$, then the two branches will arrive at the largest critical point.

The choice of $\kappa < \alpha$ close enough to α requires only satisfying two conditions: first, it must be between the critical lines under study and, second, it must be in the interval (α',α), where α' is the largest real root of $\mathbf{S}_0(x,\gamma)$ strictly smaller than α.

Fig. 3, left, shows the amount of information needed to determine the topology of the cutcurve after all the relevant points have been determined and, right, how to apply the previously presented rules to connect the branches of the cutcurve. Finally, we use the intersection points between the cutcurve and the silhouette curves to obtain the final shape that we are looking for (some branches or parts of them may be removed).

Each segment representing a branch may be discretized if required: the number of points to use will depend on the distance between the points to be connected. The points to be computed will always be far from the critical points to avoid numerical problems.

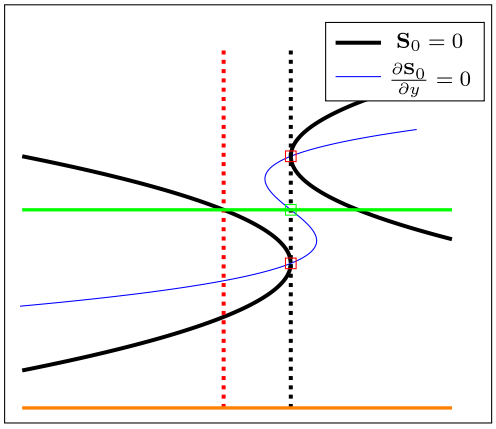

Fig. 2. Red squares: (α, β_1) and (α, β_2). Green square: (α, γ). To the left of the critical line (dotted and black), the two real branches of the cutcurve go to the smallest critical point, because the two real roots of $\mathbf{S}_0(\kappa, y) = 0$ are smaller than γ. κ is any x-value between the red and black dotted lines. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

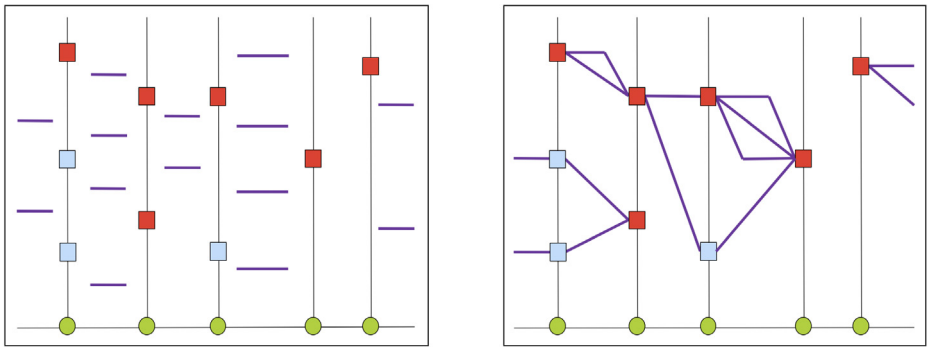

Fig. 3. Left: computed information (critical points: red, their projections: green, regular points in the critical lines: blue and branches: violet). Right: branch connection. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

5. Experimentation

This section presents the experimentation performed together with some examples showing how to analyze the intersection curve between two quadrics by using the results presented in the previous sections. In all examples, two polynomials f and g will define two quadrics $\mathcal{E}_1$ and $\mathcal{E}_2$ whose intersection curve is to be analyzed. In these examples,

$$\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2} = \{(x, y) \in \mathbb{R}^2 \colon \Delta_{\mathcal{E}_1}(x, y) \geq 0,\ \ \Delta_{\mathcal{E}_2}(x, y) \geq 0\}$$

defines the region where the cutcurve is found. Typically, the lifting of the cutcurve is performed by using $g(x, y, z) - f(x, y, z)$. When one of the singular points of the cutcurve cannot be lifted by using $g(x, y, z) - f(x, y, z)$, we use $g(x, y, z)$ or $f(x, y, z)$ (for those singular points outside the line $p_1(x, y) = q_1(x, y)$, we can also use $g(x, y, z) - f(x, y, z)$ for performing the lifting).

The process will always be as follows:

1. Compute the polynomial $\mathbf{S}_0(x, y)$:
   - When $\mathbf{S}_0(x, y)$ factors such that the cutcurve components can be parameterized either rationally or involving square roots, then this parameterization will be lifted to $\mathbb{R}^3$ to obtain the searched intersection curve.
   - When $\mathbf{S}_0(x, y)$ brings multiple factors, then the cutcurve is a conic and a line, a conic, two different lines or one line (always inside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$).

   In these two particular cases, the next steps may be used to determine what happens with the cutcurve singularities or to determine the intervals of definition for the parameterizations obtained.
2. Compute the description of the region $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$ where the cutcurve is found, which is bounded by two conics. The polynomial inequalities $\Delta_{\mathcal{E}_1}(x, y) \geq 0$ and $\Delta_{\mathcal{E}_2}(x, y) \geq 0$ will be used to determine which points on $\mathbf{S}_0(x, y) = 0$ we will consider (the cutcurve).

3. Compute the singular points of the cutcurve in the line $p_1(x, y) = q_1(x, y)$ when they exist; this involves just solving a quadratic univariate equation (as described in Section 4.3). Their lifting is performed by using f or g. This step requires solving at most three univariate quadratic equations.
4. Compute the critical points of the curve $\mathbf{S}_0(x, y) = 0$ and detect the singular points not in the line $p_1(x, y) = q_1(x, y)$ by using the formulae in Sections 4.2, 4.4 and 4.5 (by solving only one degree ≤ 8 univariate equation and several univariate quadratic equations). The lifting of these points is performed by using $g(x, y, z) - f(x, y, z)$.
5. Compute the regular points of the curve $\mathbf{S}_0(x, y) = 0$ in the critical lines by using the formulae in Section 4.2 (by solving only several univariate quadratic equations) and how the different branches connect the points in every pair of consecutive critical lines by using the approach shown in Section 4.7. The lifting of these regular points is performed by using $g(x, y, z) - f(x, y, z)$.
6. Compute the points of the cutcurve in the silhouette curves by using Proposition 4.11 (this step requires intersecting two pairs of conics). The lifting of these regular points is performed by using $g(x, y, z) - f(x, y, z)$.
7. Use the previously computed information to determine the topology of the cutcurve.
8. Compute the branches of the cutcurve (always inside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$) by using closed formulae involving radicals or discretizing them. In the second case, this involves numerically computing the solutions of the univariate equations $\mathbf{S}_0(\tau, y)$ (at most degree four and always without multiple roots) for several values of τ. The discretization is always led by the intervals defined by two consecutive critical lines.
9. Obtain the intersection curve of the two considered quadrics by lifting the branches of the cutcurve by using $g(x, y, z) - f(x, y, z)$ and including all the points computed in the previous steps.

Next, we present four examples showing several situations that may arise when using the tools previously described. The first two examples show cases in which a parameterization in closed form for the intersection curve is determined. In the remaining examples, the lifting of the cutcurve is determined after discretizing the regular branches of the cutcurve (always inside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$).

In the second, third and fourth, examples, for all figures, the following rules apply:

- left: the cutcurve is drawn in black, silhouette curves are drawn in blue, the line $p_1(x, y) = q_1(x, y)$ is dotted, and the admissible region is the darker region.
- center and/or right: the intersection curve between the considered quadrics is shown.

All computations were performed with `Maple`.

Example 5.1. We continue here with Example 3.3. In this case, the cutcurve is the ellipse $2x^2 - xy + 2y^2 - 2x - 7 = 0$ that can be parameterized by using square roots

$$y = \frac{x \pm \sqrt{-15x^2 + 16x + 56}}{4}$$

or rationally:

$$x(t) = \frac{-t^2 + 22t + 5}{3\,t^2 - 3t + 12}, \quad y(t) = \frac{5t^2 + 10t - 22}{3t^2 - 3t + 12}$$

(by using the `parametrization[algcurves]` function in `Maple`). The intersection points between the cutcurve and each silhouette curve can be computed by using Proposition 4.11, requiring determining the intersection of $xy + 2x = 7$ with each $\Delta_{\mathcal{E}_i}(x, y)$: this provides two points simultaneously in the cutcurve and in both silhouette curves.

In the first case, the presented parameterization is valid only in the interval

$$\left[\frac{8 - 2\sqrt{226}}{15}, \frac{8 + 2\sqrt{226}}{15}\right]$$

but, to define the cutcurve, from this closed interval, we must exclude the open interval (s_1, s_2) with $s_1 < s_2$, the unique two real roots of the polynomial $s^4 - 3s^2 - 28s + 49$. In the second case, the cutcurve is defined when t is outside the open interval (t_1, t_2), with $t_1 < t_2$ the unique two real roots of the polynomial $37t^4 - 182t^3 + 213t^2 - 284t + 499$.

Fig. 1 shows both the cutcurve and its lifting to the intersection curve of the two considered quadrics.

Example 5.2. Let f and g be the polynomials

$$f(x, y, z) = z^2 + xz + y \qquad g(x, y, z) = z^2 + yz + x$$

defining two hyperbolic paraboloids, $\mathcal{E}_1$ and $\mathcal{E}_2$, whose intersection curve is to be computed. In this case, we have

$$\mathbf{S}_0(x, y) = (x - y)^2(x + y + 1)$$

and $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2} = \{(x, y) \in \mathbb{R}^2 \colon x^2 - 4y \geq 0,\ y^2 - 4x \geq 0\}$. All points in the line $p_1(x, y) = q_1(x, y)$ belong to the cutcurve, and they are singular. The lifting of $\Pi(\mathcal{E}_1 \cap \mathcal{E}_2)$ outside the singular points of the cutcurve will be performed by using

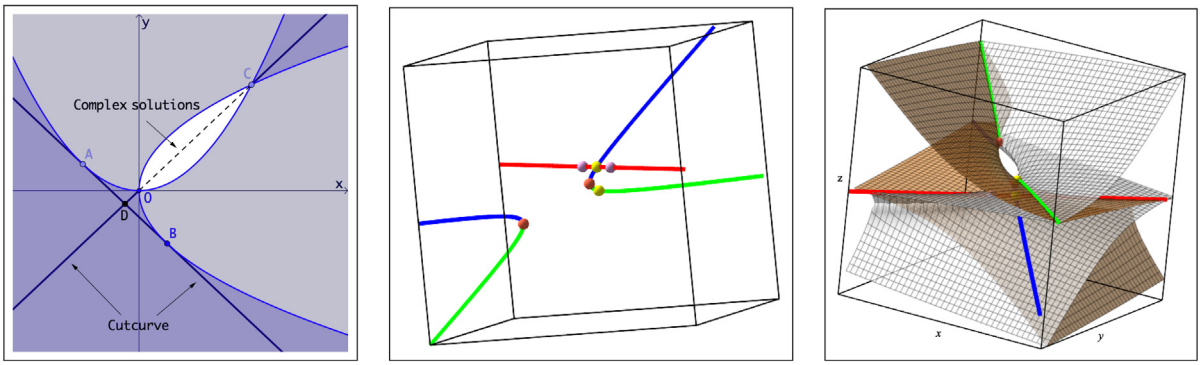

Fig. 4. Left: cutcurve (black) and silhouette curves (blue). Center: intersection curve (each component with a different color), where yellow points correspond to the lifting of D (the two tangential intersection points), orange points correspond to the lifting of O and C, and pink points correspond to the lifting of A and B. Right: intersection curve with the two quadrics. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$g(x, y, z) - f(x, y, z)$:

$$z = \frac{x-y}{x-y} = 1.$$

Singular points of the cutcurve will be lifted by using $f(x, y, z)$ or $g(x, y, z)$.

The intersection curve between $\mathcal{E}_1$ and $\mathcal{E}_2$ in this case is very easy to determine, because the two components of the cutcurve are lines allowing y to be described rationally in terms of x. Thus, the parameterization of the intersection curve is given by the following three components:

- For $x \in]-\infty, 0] \cup [4, +\infty[$: $\left(x, x, -\frac{x}{2} \pm \frac{\sqrt{x^2-4x}}{2}\right)$.
- For $x \in \mathbb{R}$: $(x, -x-1, 1)$.

The intervals for the first two components are determined either by analyzing the radical expressions and solving the corresponding (univariate) polynomial inequalities or by determining the intersection of the cutcurve with $\Delta_{\mathcal{E}_1}(x, y) = 0$ and with $\Delta_{\mathcal{E}_2}(x, y) = 0$. This is accomplished by using Proposition 4.11:

1. Points $O = (0, 0)$, $B = (1, -2)$ and $C = (4, 4)$ belong to $\mathbf{S}_0(x, y) = 0$ and $\Delta_{\mathcal{E}_1}(x, y) = 0$.
2. Points $A = (-2, 1)$, $O = (0, 0)$ and $C = (4, 4)$ belong to $\mathbf{S}_0(x, y) = 0$ and $\Delta_{\mathcal{E}_2}(x, y) = 0$.

Note that points O and C belong to $\mathbf{S}_0(x, y) = 0$, $\Delta_{\mathcal{E}_1}(x, y) = 0$ and $\Delta_{\mathcal{E}_2}(x, y) = 0$: according to Corollary 4.12, they belong to the line $x - y = 0$ too. Finally, there is one special singular point, $D = (-1/2, -1/2)$, the unique critical point of the squarefree part of $\mathbf{S}_0(x, y) = 0$. The lifting of D produces the only two tangential intersection points: $(-1/2, -1/2, 1)$ and $(-1/2, -1/2, -1/2)$. And the projection of these two (tangential intersection) points belongs to the line $p_1(x, y) = q_1(x, y)$. The cutcurve, the intersection curve and the two considered quadrics can be found in Fig. 4.

By using the QI online computation server (available at https://gamble.loria.fr/qi/server/), we obtain a parameterization of the intersection curve without involving radicals. Instead, we have shown how to obtain easily a parameterization of the intersection curve (involving radicals) by solving several degree two equations.

Example 5.3. Let f and g be the polynomials

$$f(x, y, z) = z^2 + (y - 2x - 1)z - x^2 - y^2 - xy + x - y + 1$$
$$g(x, y, z) = z^2 + (x - y)z + x^2 + y^2 - xy - 2x + y - 5$$

defining one hyperboloid of one sheet, $\mathcal{E}_1$, and one ellipsoid, $\mathcal{E}_2$, whose intersection curve is to be computed.

From Theorem 4.3 and Lemma 4.4, the singular points of the cutcurve in the line $p_1(x, y) = q_1(x, y)$ are determined by intersecting this line with the conic $p_0(x, y) = q_0(x, y)$. They are

$$A = \left(-\frac{3}{13} - \frac{3\sqrt{14}}{13}, \frac{2}{13} - \frac{9\sqrt{14}}{26}\right) \qquad B = \left(-\frac{3}{13} + \frac{3\sqrt{14}}{13}, \frac{2}{13} + \frac{9\sqrt{14}}{26}\right)$$

and they can be lifted by using $g(x, y, z) = 0$ or $f(x, y, z) = 0$, producing four points in the intersection curve. There are no more singular points of the cutcurve: in this case, after we remove from $\sigma_0(x)$ the degree 4 factor giving the projection

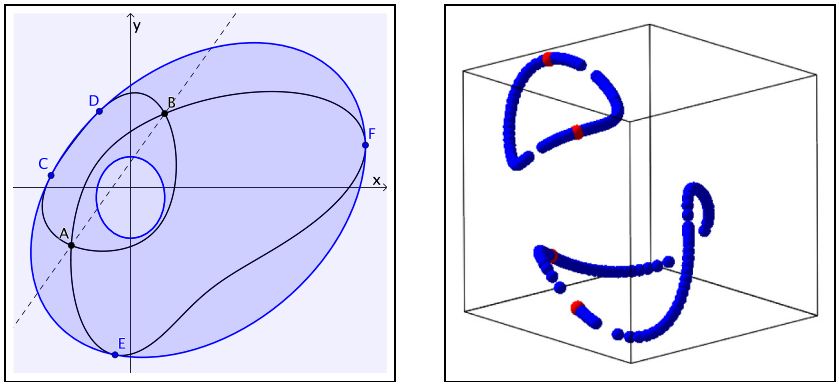

Fig. 5. Left: cutcurve (black), silhouette curves (blue). Right: intersection curve. Red points correspond to the lifting of A and B. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of A and B, we obtain a degree 8 polynomial with four different real roots providing four critical (and non-singular) points of the cutcurve.

Finally, we use Proposition 4.11 to determine the intersection of the cutcurve with the silhouette curves:

1. There are no common points to $\mathbf{S}_0(x, y) = 0$ and $\Delta_{\mathcal{E}_1}(x, y) = 0$.
2. Points $C = (-1.468654, 0.233082)$, $D = (-0.575622, 1.494633)$, $E = (-0.285566, -3.292475)$ and $F = (4.341889, 0.829820)$ belong to $\mathbf{S}_0(x, y) = 0$ and $\Delta_{\mathcal{E}_2}(x, y) = 0$.

The lifting of $\Pi(\mathcal{E}_1 \cap \mathcal{E}_2)$, outside the singular points of the cutcurve will be performed by using $g(x, y, z) - f(x, y, z)$. The cutcurve and the intersection curve (discretized) can be found in Fig. 5.

The last example shows a situation in which the introduced tools are especially useful for determining the properties of the intersection curve of the two considered quadrics.

Example 5.4. Let f and g be the polynomials

$$f(x, y, z) = z^2 + \left(\frac{-2x + 2y}{3}\right) z + \frac{x^2 + y^2 - 1}{3}$$
$$g(x, y, z) = z^2 + \left(\frac{-2x + 24y - 2}{17}\right) z + \frac{x^2 + 12y^2 + 2x - 3}{17}$$

defining two ellipsoids, $\mathcal{E}_1$ and $\mathcal{E}_2$, whose intersection curve is to be computed.

From Theorem 4.3 and Lemma 4.4, the singular points of the cutcurve in the line $p_1(x, y) = q_1(x, y)$ are determined by intersecting this line with the conic $p_0(x, y) = q_0(x, y)$. They are

$$A = \left(\frac{3}{14} - \frac{11\sqrt{95}}{70}, \frac{11\sqrt{95}}{95}\right) \qquad C = \left(\frac{3}{14} + \frac{11\sqrt{95}}{70}, -\frac{11\sqrt{95}}{95}\right)$$

but they are outside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$, and they will not be lifted.

To compute the singular points of the cutcurve outside the line $p_1(x, y) = q_1(x, y)$, we apply the formulae introduced in Sections 4.2 and 4.4. We compute $\sigma_0(x)$ and remove the factor providing A and C (see Section 4.5):

$$\frac{\sigma_0(x)}{(70x^2 - 30x - 161)^2} = (9x^2 + 6x - 7)(x - 1)^2(22x^2 - 4x - 43)^2$$

Analyzing the real roots of this polynomial allows us to conclude that the cutcurve has a third singular (and isolated) point $B = (1, 0)$, which is inside $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$ but not in the line $p_1(x, y) = q_1(x, y)$. B is the projection of a tangential intersection point of $\mathcal{E}_1$ and $\mathcal{E}_2$, $(1, 0, 0)$, because it is not a singular point of $\mathcal{E}_1$ and $\mathcal{E}_2$ (as seen in Theorem 4.6).

Finally, we use Proposition 4.11 to determine the intersection of the cutcurve with the silhouette curves:

1. Points $D = (-1.310086292, 1.116297338)$ and $E = (-1.032926046, -0.320076179)$ belong to $\mathbf{S}_0(x, y) = 0$ and $\Delta_{\mathcal{E}_1}(x, y) = 0$.
2. Points $F = (-1.310059433, 1.116308957)$ and $G = (0.4229961827, -1.105788551)$ belong to $\mathbf{S}_0(x, y) = 0$ and $\Delta_{\mathcal{E}_2}(x, y) = 0$.

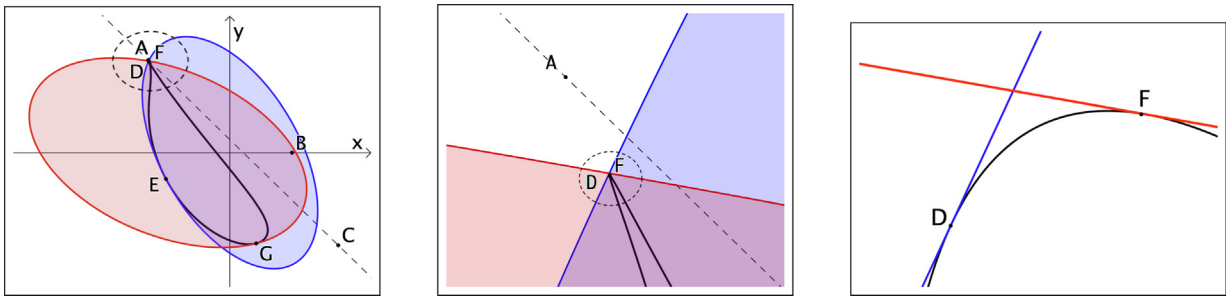

Fig. 6. Cutcurve (black) and silhouette curves (red and blue) showing the locations of points A, F and D (left). The locations of points A, F and D with respect to the cutcurve and the silhouette curves (center and right). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

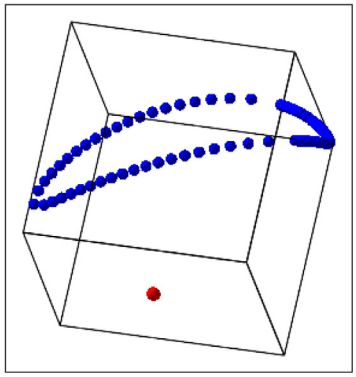

Fig. 7. Two ellipsoids with a curve and an isolated point (tangent to both ellipsoids) in common.

Fig. 6 (left) shows the locations of all these points with respect to the cutcurve and the silhouette curves. To determine the relative position of the points A, F and D with respect to these three curves, we use the results introduced at the end of Section 4.6 (see, in particular, Corollary 4.12). Fig. 6 (center and right) shows in detail what is happening in that area.

Point A is outside the region $\mathcal{A}_{\mathcal{E}_1,\mathcal{E}_2}$ and does not play a role in computing the intersection curve between the considered quadrics. Points F and D belong to the intersection between the cutcurve and one of the silhouette curves. Corollary 4.12 allows us to conclude that the intersection between the cutcurve and the two silhouette curves is empty and not A, as Fig. 6 (left) might suggest.

The lifting of the regular points of the cutcurve and of the point B will be performed by using $g(x, y, z) - f(x, y, z)$. The intersection curve of the two considered quadrics can be found in Fig. 7.

The tools introduced here have been fully implemented in `Maple`. Fig. 8 presents the output of our `Maple` implementation when computing the intersection curve for three different pairs of quadrics. Third example shows a concrete case in which the intersection curve is discretized.

Procedure `par` attempts to compute a closed form for some of the components of the intersection curve and, when involving radicals, determines those intervals in which such a description can be evaluated. Procedure `sing` computes three lists of points: the first one contains the tangential and singular intersection points, if any, the second one contains the lifting to the intersection curve of the singular points of the cutcurve in the line $p_1(x, y) = q_1(x, y)$, if any, and the third one contains a discretization for the components of the intersection curve without a parameterization in closed form directly available.

Table 1 shows the behavior of this implementation when applied to a database of 50 examples, most taken from the QI online computation server available at https://gamble.loria.fr/qi/server/. In the Annex, the equations of each pair of quadrics used can be found. The meaning of the columns in Table 1 is as follows:

- The third column reflects whether the intersection curve has been discretized.
- The fourth column shows the number of points computed when the intersection curve has been discretized.
- The fifth column shows whether tangential or singular intersection points exist whose projection is outside the line $p_1(x, y) = q_1(x, y)$.

```
par((-16*x^2+8*x*z+8*y^2-8*z^2+48*x-32*y-24*z)*(-1/8), (-12*x*z+4*y^2+2*z^2+8*x-16*y-4*z)*(1/2));
```

$$\left[\left[\left[x, 2+2\sqrt{x^2+1}, 2x\right], \left[x, 2-2\sqrt{x^2+1}, 2x\right], \left[x, 2+\frac{2\sqrt{4x^2-10x+4}}{3}, \frac{2x}{3}-\frac{4}{3}\right], \left[x, 2-\frac{2\sqrt{4x^2-10x+4}}{3}, \frac{2x}{3}-\frac{4}{3}\right], \left[\left[-\frac{1}{2}, \frac{1}{2}\right], [2, 3]\right]\right]\right]$$

```
sing((-16*x^2+8*x*z+8*y^2-8*z^2+48*x-32*y-24*z)*(-1/8), (-12*x*z+4*y^2+2*z^2+8*x-16*y-4*z)*(1/2));
```

$$\left[[\,], \left[\left[-1, 2-2\sqrt{2}, -2\right], \left[-1, 2+2\sqrt{2}, -2\right]\right], [\,]\right]$$

```
> sing(32*x^2-48*x*y+64*x*z+14*y^2-34*z^2+32*x-8*y-72*z-16, 48*x^2-48*x*y+24*x*z+9*y^2-9*z^2-16*x+12*y-12*z)
```

$$\left[\left[[0.3819660113, 0.3519093630, -0.1573786518], [0.6666666667, 2.000000000, -0.6666666672], [1.500000000, 2.000000000, -6.629999999\,10^{-10}], [2.618033989, 6.314757311, 2.824045318]\right], \left[\left[-1, -\frac{14}{3}+\frac{4\sqrt{5}}{3}, -\frac{10}{3}+\frac{4\sqrt{5}}{3}\right], \left[-1, -\frac{14}{3}+\frac{4\sqrt{5}}{3}, -\frac{2}{3}-\frac{4\sqrt{5}}{3}\right], \left[-1, -\frac{14}{3}-\frac{4\sqrt{5}}{3}, -\frac{2}{3}+\frac{4\sqrt{5}}{3}\right], \left[-1, -\frac{14}{3}-\frac{4\sqrt{5}}{3}, -\frac{10}{3}-\frac{4\sqrt{5}}{3}\right]\right], [\,]\right]$$

```
> par((32*x^2-48*x*y+64*x*z+14*y^2-34*z^2+32*x-8*y-72*z-16, 48*x^2-48*x*y+24*x*z+9*y^2-9*z^2-16*x+12*y-12*z)
```

$$\left[\left[\left[x, 4x-\frac{2}{3}+\frac{4\sqrt{5}\,x}{5}-\frac{8\sqrt{5}}{15}, -\frac{2}{3}+\frac{(-12x+8)\sqrt{5}}{15}\right], \left[x, 4x-\frac{2}{3}-\frac{4\sqrt{5}\,x}{5}+\frac{8\sqrt{5}}{15}, -\frac{2}{3}+\frac{(12x-8)\sqrt{5}}{15}\right], \left[x, \frac{8x}{3}-2+\frac{8\sqrt{5}\,x}{15}-\frac{4\sqrt{5}}{5}, \frac{(8x-12)\sqrt{5}}{15}+\frac{4x}{3}-2\right], \left[x, \frac{8x}{3}-2-\frac{8\sqrt{5}\,x}{15}+\frac{4\sqrt{5}}{5}, \frac{(-8x+12)\sqrt{5}}{15}+\frac{4x}{3}-2\right]\right]\right]$$

```
> par(48*x^2-96*x*y+8*x*z+44*y^2-4*y*z+4*z^2+80*x-72*y+16*z+48, -24*x^2+52*x*z+2*y^2-10*y*z-18*z^2+40*x-4*
  y-40*z-24)
```

$$[[\,]]$$

```
> sing(48*x^2-96*x*y+8*x*z+44*y^2-4*y*z+4*z^2+80*x-72*y+16*z+48, -24*x^2+52*x*z+2*y^2-10*y*z-18*z^2+40*
  x-4*y-40*z-24)
```

[ ], [ ], [[−4.344526811, −4.048555471, −2.388763429], [−4.344526811, −3.982907043, −2.388004965], [−4.297136261, −4.008781568, −2.368323547], [−4.297136261, −3.918472459, −2.367251752], [−4.249745710, −3.965228355, −2.347843176], [−4.249745710, −3.857809940, −2.346534136], [−4.202355159, −3.919538652, −2.327339500], [−4.202355159, −3.799277195, −2.325835208], [−4.154964609, −3.872346607, −2.306818594], [−4.154964609, −3.742239614, −2.305148742], [−4.107574058, −3.823974265, −2.286283225], [−4.107574058, −3.686374787, −2.284471828], [−4.060183507, −3.774615326, −2.265735067], [−4.060183507, −3.631489666, −2.263803136], [−4.012792957, −3.724381755, −2.245174690], [−4.012792957, −3.577471891, −2.243142028], [−3.965402406, −3.673342909, −2.224602254], [−3.965402406, −3.524252141, −2.222488401], [−3.918011856, −3.621533607, −2.204017390], [−3.918011856, −3.471795200, −2.201842483], [−3.870621305, −3.568968925, −2.183419630], [−3.870621305, −3.420086052, −2.181204924], [−3.823230754, −3.515641398, −2.162808175], [−3.823230754, −3.369132566, −2.160576723], [−3.775840204, −3.461512538, −2.142181529], [−3.775840204, −3.318972671, −2.139959226], [−3.728449653, −3.406517395, −2.121537836], [−3.728449653, −3.269671349, −2.119354412], [−3.681059102, −3.350549380, −2.100874519], [−3.681059102, −3.221335421, −2.098765127], [−3.633668552, −3.293422865, −2.080187297], [−3.633668552, −3.174150194, −2.078195539], [−3.586278001, −3.234820415, −2.059469443], [−3.586278001, −3.128433026, −2.057652492], [−3.538887450, −3.174121845, −2.038708540], [−3.538887450, −3.084804478, −2.037148758], [−3.491496900, −3.109712611, −2.017873131], [−3.491496900, −3.044878505, −2.016715622], [−3.444106349, −3.025333350, −1.996662147], [−3.444106349, −3.024914767, −1.996654509]]]

Fig. 8. Three examples of using `Maple` to compute the intersection curve of two quadrics.

The performance of these tools has been compared with the `Maple` `intersectplot` command. Next, we discuss the advantages of our algorithm compared with the `intersectplot` command computing the intersection curve of two surfaces in $\mathbb{R}^3$:

1. The first issue is that the `intersectplot` command works locally, requiring the region where the intersection curve is to be computed to be decided in advance. Thus, some connected components of the intersection curve may be missing. Our tools do not miss any connected components of the intersection curve.
2. The second issue is that the `intersectplot` command may miss some points when the intersection curve contains a finite set of isolated points.
3. The third issue concerns the quality of the output produced by the `intersectplot` command when the box in the input does not adequately fit the intersection curve: depending on the size of the box, the quality and accuracy of the output change drastically.

The practical behavior of these tools is quite good; they are very easy to use and provide several improvements, especially when the intersection curve is to be discretized. At this moment this is made by solving the equation $\mathbf{S}_0(\tau, y) = 0$ for τ belonging to the different intervals provided by the critical lines of the curve $\mathbf{S}_0(x, y) = 0$ as many times as needed. This implementation is available upon request.

Remark 5.5. The four examples included here and the reported experimentation consider two quadrics whose coefficients are rational numbers. If that is not the case, i.e., if the coefficients of the quadrics equations are floating point real numbers, then first we need to fix a tolerance ϵ to be used when deciding whether two numbers are equal. All results included here are still true, but, to avoid the numerical problems arising when computing determinants, we should proceed as in [36]. In particular:

- To compute the critical lines, we evaluate $\mathbf{S}_0(x, y)$ by using the presentation in (3) or in Lemma 4.10: this implies that the critical lines are obtained after solving a generalized eigenvalue problem generated from the Bezoutian matrix of $\mathbf{S}_0(x, y)$ and its partial derivative with respect to y (with y the variable to be eliminated).
- To compute the y-coordinates of the critical points in $x = \alpha$ requires determining the nullspace of the previously computed Bezoutian matrix after x is replaced by α (which is a 4×4 matrix).

Table 1
Results from applying the `Maple` implementation to the quadrics in the annex.

Example	Time	Discretization	Number of Points	Tangential/Singular intersections
1	0.395	yes	40	no
2	1.295	yes	242	no
3	1.380	yes	244	no
4	0.313	no		no
5	1.586	yes	476	no
6	1.502	yes	472	no
7	1.338	yes	362	no
8	1.290	yes	202	yes
9	0.822	yes	80	no
10	3.791	yes	840	no
11	0.193	no		no
12	3.387	yes	762	no
13	0.236	no		no
14	0.353	no		no
15	0.357	no		no
16	0.292	no		no
17	0.330	no		no
18	0.294	yes	40	no
19	0.224	no		no
20	0.372	no		no
21	3.114	yes	1114	no
22	0.215	no		no
23	0.247	no		no
24	0.616	no		no
25	0.265	no		no
26	0.334	no		no
27	3.541	yes	558	no
28	0.296	no		no
29	1.664	yes	472	no
30	0.181	no		no
31	0.143	no		no
32	0.200	no		no
33	0.290	no		no
34	0.234	no		no
35	0.273	no		no
36	0.329	no		no
37	0.351	no		no
38	0.237	no		no
39	0.242	no		no
40	0.234	no		yes
41	0.272	no		yes
42	0.200	no		yes
43	0.332	no		no
44	0.899	yes	160	no
45	0.310	no		no
46	0.304	no		yes
47	1.165	no		yes
48	5.835	yes	1574	yes
49	6.542	yes	1894	yes
50	1.195	yes	120	yes

In this way, computing the roots of determinants of polynomial matrices is replaced by solving generalized eigenvalue problems, and subresultants are replaced by nullspace computations of Bezoutian matrices. Even if the degree of $\mathbf{S}_0(x, y)$ is small and makes this problem easier, the numerical issues discussed in [36] (Section 8) are still valid. Moreover, the topology of the cutcurve depends on the value of the initially fixed tolerance ϵ.

6. Conclusions

We have introduced a new approach to the analysis of the intersection curve between two quadrics. The introduced tools provide a detailed analysis of the cutcurve, its critical and singular points, and its relationship with the silhouette curves together with the use of a uniform way to lift the cutcurve to the intersection curve between the considered quadrics.

This approach is not intended to classify the intersection curve between the considered quadrics. Its main goal is to directly produce a topologically correct description of the intersection curve. This is why in the lifting of the cutcurve, when possible, we allow the use of radicals, or we rely on the discretization of the branches of the cutcurve (uniquely determined by the points computed in that curve).

The algorithm has been fully implemented in `Maple` and shows a very good practical behavior.

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Annex

Example	Quadrics
Example 1	$z^2+(2x-y+4)z+12x^2-24xy+11y^2+20x-18y+12$ $z^2+\left(-\frac{26x}{9}+5/9y+\frac{20}{9}\right)z+4/3x^2-1/9y^2-\frac{20x}{9}+2/9y+4/3$
Example 2	$z^2+(-x+3)z+x^2-3/2xy+1/2y^2-x-y/2+2$ $z^2-8x^2+18xy-4xz-7y^2-12x+10y-4$
Example 3	$z^2+\left(-8/5x+\frac{12}{5}\right)z-\frac{12x^2}{5}+\frac{28xy}{5}-\frac{13y^2}{5}-\frac{32x}{5}+\frac{24y}{5}-4/5$ $z^2+\left(-\frac{16x}{25}+\frac{84}{25}\right)z-\frac{28x^2}{25}+\frac{12xy}{5}-\frac{27y^2}{25}-\frac{112x}{25}+\frac{48y}{25}+\frac{52}{25}$
Example 4	$z^2+\left(-\frac{88x}{43}+\frac{84}{43}\right)z+\frac{60x^2}{43}-\frac{28xy}{43}+\frac{13y^2}{43}-\frac{64x}{43}-\frac{24y}{43}+\frac{52}{43}$ $z^2+\left(-\frac{80x}{23}+\frac{12}{23}\right)z+\frac{76x^2}{23}-\frac{60xy}{23}+\frac{27y^2}{23}+\frac{16x}{23}-\frac{48y}{23}-\frac{4}{23}$
Example 5	$z^2+(-2x+2)z+3/2xy-1/2y^2-3x+y/2+1$ $z^2+\left(-\frac{16x}{7}+\frac{12}{7}\right)z+\frac{10xy}{7}-3/7y^2-\frac{20x}{7}+2/7y+4/7$
Example 6	$z^2+(-2x+2)z+3/2xy-1/2y^2-3x+y/2+1$ $z^2+\left(-\frac{32x}{15}+\frac{28}{15}\right)z+\frac{8x^2}{15}+2/3xy-1/5y^2-\frac{12x}{5}+2/15y+4/5$
Example 7	$z^2+(-2x+2)z+3/2xy-1/2y^2-3x+y/2+1$ $z^2+\left(-\frac{16x}{9}+\frac{20}{9}\right)z+\frac{10xy}{9}-1/3y^2-\frac{20x}{9}+2/9y+4/3$
Example 8	$z^2+(-3x+2y+1)z+4x^2-4xy+1/2y^2-2x+2y$ $z^2+\left(-\frac{30x}{13}+\frac{4y}{13}+\frac{22}{13}\right)z+\frac{16x^2}{13}-\frac{4xy}{13}-\frac{28x}{13}+\frac{4y}{13}+\frac{8}{13}$
Example 9	$z^2+(-8/3x+4/3)z+8/3x^2-2xy+2/3y^2-4/3x-2/3y$ $z^2+\left(-\frac{12x}{7}+\frac{16}{7}\right)z+\frac{16x^2}{7}-2xy+4/7y^2-\frac{12x}{7}-2/7y+\frac{8}{7}$
Example 10	$z^2+(-x+3)z-2x^2+3xy-y^2-4x+y+2$ $z^2+(-2x+2)z+\frac{4x^2}{13}+\frac{10xy}{13}-3/13y^2-\frac{32x}{13}+2/13y+\frac{12}{13}$
Example 11	$z^2+(-2x+2)z+3/2xy-1/2y^2-3x+y/2+1$ $z^2+4z+8x^2-10xy+3y^2+4x-2y+4$
Example 12	$z^2+(-4x+2y+2)z+2x^2-xy-6x+3y$ $z^2+(-2x-y+6)z-2x^2+3xy-10x+y+4$
Example 13	$z^2+(-2x+2)z+3x^2-3xy+5/4y^2-2y+2$ $z^2+\left(-\frac{12x}{5}+8/5\right)z+2x^2-6/5xy+1/2y^2-8/5x-4/5y+4/5$
Example 14	$z^2+(-2x+2)z+x^2-xy+3/4y^2-2y+2$ $z^2+(-4/3x+8/3)z+2/3x^2+2/3xy-1/2y^2-8/3x+4/3y+4/3$
Example 15	$z^2+(-2x+2)z+1/2y^2-2y+2$ $z^2+(-4/3x+8/3)z+4/3x^2-1/3y^2-8/3x+4/3y+4/3$
Example 16	$z^2+(-8/3x+4/3)z-4/3x^2+4xy-5/3y^2-16/3x+8/3y-4/3$ $z^2+\left(-\frac{16x}{11}+\frac{28}{11}\right)z-4/11x^2+\frac{12xy}{11}-\frac{5y^2}{11}-\frac{32x}{11}+\frac{8y}{11}+\frac{12}{11}$
Example 17	$z^2+\left(-\frac{32x}{15}+\frac{28}{15}\right)z+\frac{8x^2}{15}+4/5xy-1/3y^2-8/3x+\frac{8y}{15}+\frac{8}{15}$ $z^2+\left(-\frac{16x}{13}-\frac{32y}{39}+\frac{36}{13}\right)z+\frac{8x^2}{39}+\frac{28xy}{39}+1/13y^2-\frac{24x}{13}-\frac{40y}{39}+\frac{24}{13}$
Example 18	$z^2+(-8/3x+4/3)z-4/3x^2+4xy-5/3y^2-16/3x+8/3y-4/3$ $z^2+\left(-\frac{24x}{7}+\frac{32y}{21}+4/7\right)z+\frac{52x^2}{21}-4/3xy-1/7y^2-\frac{16x}{7}+\frac{40y}{21}-4/7$
Example 19	$z^2+(-8/3x+4/3)z+4/3xy-1/3y^2-8/3x$ $z^2+\left(-8/5x+\frac{12}{5}\right)z+\frac{12xy}{5}-7/5y^2-\frac{24x}{5}+\frac{16y}{5}$
Example 20	$z^2+\left(-\frac{24x}{11}+\frac{20}{11}\right)z+\frac{8x^2}{11}+4/11xy-1/11y^2-\frac{24x}{11}+\frac{8}{11}$ $z^2+(-8/3x+4/3)z+8/3x^2-4xy+7/3y^2+8/3x-16/3y+8/3$
Example 21	$z^2+\left(-\frac{24x}{11}+\frac{20}{11}\right)z+\frac{8x^2}{11}+4/11xy-1/11y^2-\frac{24x}{11}+\frac{8}{11}$ $z^2+\left(-\frac{504x}{253}+\frac{508}{253}\right)z+\frac{248x^2}{253}+\frac{12xy}{253}-\frac{7y^2}{253}-\frac{520x}{253}+\frac{16y}{253}+\frac{248}{253}$

Example	Quadrics
Example 22	$z^2 - 4xz + 2xy - 1/2\,y^2 - 4x - 2$ $z^2 - 4xz - 2xy + 5/2\,y^2 + 4x - 8y + 2$
Example 23	$z^2 + \left(-\frac{24x}{11} + \frac{20}{11}\right)z + \frac{8x^2}{11} + 4/11\,xy - 1/11\,y^2 - \frac{24x}{11} + \frac{8}{11}$ $z^2 + \left(-\frac{24x}{11} + \frac{20}{11}\right)z + \frac{8x^2}{11} - 4/11\,xy + \frac{5y^2}{11} - \frac{8x}{11} - \frac{16y}{11} + \frac{16}{11}$
Example 24	$z^2 + \left(-\frac{24x}{11} + \frac{20}{11}\right)z + \frac{8x^2}{11} + 4/11\,xy - 1/11\,y^2 - \frac{24x}{11} + \frac{8}{11}$ $z^2 + \left(-\frac{24x}{11} + \frac{20}{11}\right)z - \frac{248x^2}{11} + \frac{252xy}{11} - \frac{59y^2}{11} - \frac{8x}{11} - \frac{16y}{11} + \frac{16}{11}$
Example 25	$z^2 + (-x + 3)z + 2x^2 - y^2 - 6x + 4y$ $z^2 + (-6x - 2)z + 2y^2 + 4x - 8y$
Example 26	$z^2 + \left(-\frac{88x}{43} + \frac{84}{43}\right)z + \frac{52x^2}{43} - \frac{4xy}{43} - 1/43\,y^2 - \frac{96x}{43} + \frac{8y}{43} + \frac{36}{43}$ $z^2 + \left(-\frac{32x}{17} + \frac{36}{17}\right)z + \frac{52x^2}{51} - \frac{4xy}{51} - \frac{y^2}{51} - \frac{112x}{51} + \frac{8y}{51} + \frac{52}{51}$
Example 27	$z^2 + \left(-\frac{88x}{43} + \frac{84}{43}\right)z + \frac{52x^2}{43} - \frac{4xy}{43} - 1/43\,y^2 - \frac{96x}{43} + \frac{8y}{43} + \frac{36}{43}$ $z^2 + \left(-\frac{24x}{5} - 4/5\right)z + \frac{28x^2}{5} - 4/5\,xy - 1/5\,y^2 - \frac{16x}{5} + 8/5\,y - \frac{12}{5}$
Example 28	$z^2 + 4z - 4x^2 + 6xy - 2y^2 - 4x + 2y + 4$ $z^2 + \left(-8/5\,x + \frac{12}{5}\right)z - 4/5\,x^2 + \frac{14xy}{5} - 6/5\,y^2 - 4x + 2y + 4/5$
Example 29	$z^2 + 4z - 4x^2 + 6xy - 2y^2 - 4x + 2y + 4$ $z^2 + \left(-\frac{16x}{9} + \frac{20}{9}\right)z + \frac{14xy}{9} - 2/3\,y^2 - \frac{28x}{9} + \frac{10y}{9} + \frac{8}{9}$
Example 30	$z^2 + 4z + 8x^2 - 12xy + 5y^2 + 8x - 8y + 8$ $z^2 + \left(-\frac{16x}{7} + \frac{12}{7}\right)z + \frac{12xy}{7} - 5/7\,y^2 - \frac{24x}{7} + \frac{8y}{7}$
Example 31	$z^2 + (-8/3\,x + 4/3)z + 4/3\,xy - 1/3\,y^2 - 8/3\,x$ $z^2 + (-8/3\,x + 4/3)z + 8/3\,x^2 - 4xy + 7/3\,y^2 + 8/3\,x - 16/3\,y + 8/3$
Example 32	$z^2 + \left(-\frac{24x}{11} + \frac{20}{11}\right)z + \frac{8x^2}{11} + 4/11\,xy - 1/11\,y^2 - \frac{24x}{11} + \frac{8}{11}$ $z^2 + \left(-\frac{24x}{11} + \frac{20}{11}\right)z + \frac{16x^2}{11} - \frac{12xy}{11} + \frac{7y^2}{11} - \frac{8x}{11} - \frac{16y}{11} + \frac{16}{11}$
Example 33	$z^2 + (-8/3\,x + 4/3)z + 4/3\,xy - 1/3\,y^2 - 8/3\,x$ $z^2 + (-8\,x + 16/3\,y - 4)z + 16/3\,x^2 - 4xy - 1/3\,y^2 - 8/3\,x + 16/3\,y - 16/3$
Example 34	$z^2 + \left(-\frac{24x}{11} + \frac{20}{11}\right)z + \frac{8x^2}{11} + 4/11\,xy - 1/11\,y^2 - \frac{24x}{11} + \frac{8}{11}$ $z^2 + \left(-\frac{40x}{11} + \frac{16y}{11} + 4/11\right)z + \frac{24x^2}{11} - \frac{12xy}{11} - 1/11\,y^2 - \frac{24x}{11} + \frac{16y}{11} - \frac{8}{11}$
Example 35	$z^2 + (-8/3\,x + 4/3)z + 4/3\,xy - 1/3\,y^2 - 8/3\,x$ $z^2 + (-8\,x + 16/3\,y - 4)z + 16/3\,x^2 - 4xy - 1/3\,y^2 - 8/3\,x + 16/3\,y - 16/3$
Example 36	$z^2 + \left(-\frac{24x}{11} + \frac{20}{11}\right)z + 4/11\,x^2 + \frac{8xy}{11} - 2/11\,y^2 - \frac{24x}{11} + \frac{8}{11}$ $z^2 + \left(-\frac{24x}{11} + \frac{20}{11}\right)z - \frac{12x^2}{11} + \frac{32xy}{11} - \frac{10y^2}{11} - \frac{40x}{11} + \frac{8y}{11} + \frac{8}{11}$
Example 37	$z^2 + (-5/2\,x + y/2 + 3/2)z + 2x^2 - xy + 1/4\,y^2 - 3x + y$ $z^2 + (-7/2\,x + y + 3/2)z + 2x^2 - xy + 1/4\,y^2 - 5x + 2y$
Example 38	$z^2 + (-3/2\,x - y/2 + 5/2)z + xy - 1/4\,y^2 - x - y + 2$ $z^2 + (-x/2 - y + 5/2)z + xy - 1/4\,y^2 + x - 2y + 2$
Example 39	$z^2 + (-7/4\,x - y/4 + 9/4)z + 1/2\,x^2 + 1/2\,xy - 1/8\,y^2 - 3/2\,x - y/2 + 3/2$ $z^2 + (-5/4\,x - y/2 + 9/4)z + 1/2\,x^2 + 1/2\,xy - 1/8\,y^2 - x/2 - y + 3/2$
Example 40	$z^2 + \left(-8/5\,x + \frac{12}{5}\right)z - 4/5\,x^2 + \frac{12xy}{5} - 4/5\,y^2 - \frac{16x}{5} + 4/5\,y + 8/5$ $z^2 + \left(-\frac{16x}{13} + \frac{36}{13}\right)z - \frac{20x^2}{13} + \frac{36xy}{13} - \frac{14y^2}{13} - \frac{48x}{13} + \frac{20y}{13} + \frac{16}{13}$
Example 41	$z^2 + \left(-\frac{32x}{17} + \frac{36}{17}\right)z + \frac{32xy}{17} - \frac{13y^2}{17} - \frac{64x}{17} + \frac{20y}{17} + \frac{16}{17}$ $z^2 + (-8/3\,x + 4/3)z + \frac{32x^2}{9} - \frac{8xy}{9} + 1/9\,y^2 - \frac{32x}{9} + 4/9\,y + \frac{8}{9}$
Example 42	$z^2 + \left(-\frac{24x}{13} + \frac{28}{13}\right)z + \frac{24xy}{13} - \frac{10y^2}{13} - \frac{48x}{13} + \frac{16y}{13} + \frac{12}{13}$ $z^2 + \left(-\frac{16x}{5} + 4/5\right)z + \frac{24x^2}{5} - 8/5\,xy + 2/5\,y^2 - \frac{16x}{5} + 4/5$
Example 43	$z^2 + \left(-\frac{128x}{63} + \frac{124}{63}\right)z + \frac{176x^2}{63} - \frac{8xy}{21} - \frac{13y^2}{63} - \frac{304x}{63} + \frac{76y}{63} + \frac{8}{7}$ $z^2 + \left(-\frac{136x}{71} + \frac{148}{71}\right)z + \frac{368x^2}{71} - \frac{208xy}{71} + \frac{29y^2}{71} - \frac{336x}{71} + \frac{92y}{71} + \frac{80}{71}$
Example 44	$z^2 + \left(-\frac{16x}{7} + \frac{12}{7}\right)z + \frac{16xy}{7} - 5/7\,y^2 - \frac{32x}{7} + 4/7\,y + \frac{8}{7}$ $z^2 + \left(-8/5\,x + \frac{12}{5}\right)z + \frac{16x^2}{15} + \frac{8xy}{15} - \frac{7y^2}{15} - \frac{64x}{15} + 4/3\,y + \frac{16}{15}$
Example 45	$z^2 + \left(-\frac{24x}{11} + \frac{20}{11}\right)z + \frac{24xy}{11} - \frac{48x}{11} - \frac{8y^2}{11} + \frac{8y}{11} + \frac{12}{11}$ $z^2 + \left(-\frac{32x}{19} + \frac{44}{19}\right)z + \frac{24x^2}{19} + \frac{8xy}{19} - \frac{80x}{19} - \frac{8y^2}{19} + \frac{24y}{19} + \frac{20}{19}$

Example	Quadrics
Example 46	$z^2+\left(-\frac{128x}{65}+\frac{132}{65}\right)z-\frac{64x^2}{65}+\frac{112xy}{65}-\frac{29y^2}{65}-\frac{96x}{65}+\frac{4y}{65}+\frac{48}{65}$
	$z^2+\left(-\frac{40x}{19}+\frac{36}{19}\right)z-\frac{128x^2}{57}+\frac{56xy}{19}-\frac{13y^2}{19}-\frac{64x}{57}-\frac{4y}{19}+\frac{40}{57}$
Example 47	$z^2+\left(-\frac{32x}{17}+\frac{36}{17}\right)z-\frac{16x^2}{17}+\frac{24xy}{17}-\frac{7y^2}{17}-\frac{16x}{17}+\frac{4y}{17}+\frac{8}{17}$
	$z^2+(-8/3x+4/3)z-16/3x^2+16/3xy-y^2+\frac{16x}{9}-4/3y$
Example 48	$z^2+\left(-\frac{32x}{17}+\frac{36}{17}\right)z-\frac{16x^2}{17}+\frac{24xy}{17}-\frac{7y^2}{17}-\frac{16x}{17}+\frac{4y}{17}+\frac{8}{17}$
	$z^2+(-8/3x+4/3)z-16/3x^2+16/3xy-y^2+\frac{16x}{9}-4/3y$
Example 49	$z^2+\left(-\frac{128x}{65}+\frac{132}{65}\right)z-\frac{56x^2}{13}+\frac{288xy}{65}-\frac{64y^2}{65}-\frac{16x}{65}-\frac{32y}{65}+\frac{44}{65}$
	$z^2+\left(-\frac{40x}{17}+\frac{28}{17}\right)z-\frac{16x^2}{17}+\frac{32xy}{17}-\frac{7y^2}{17}-\frac{16x}{17}-\frac{4y}{17}+\frac{8}{17}$
Example 50	$z^2+(-2/3x+2/3y)z+1/3x^2+1/3y^2-1/3$
	$z^2+\left(-2/17x+\frac{24y}{17}-2/17\right)z+1/17x^2+2/17x-\frac{3}{17}+\frac{12y^2}{17}$

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