Quadric-surface intersection curves: shape and structure

Itzhak Wilf and Yehuda Manor*

When CAD models composed of quadric surfaces are processed, it is necessary to compute their mutual curve of intersection. A method for the analysis of such intersection curves is presented. Shape description is carried out by a modification of Levin's ruled-surface parameterization scheme, guided by invariant-factors classification. The decomposition of the intersection curve is performed by factorization of the parameterization polynomials. Further analysis of these polynomials yields the graph structure of the curve. The projection of the curve for boundary visualization and evaluation is also described. A full implementation for all nonplanar intersections is included.

Keywords: quadrics, intersection curves

Quadric surfaces (i.e. algebraic surfaces of degree 2) are frequently used in mechanical CAD/CAM. Being the simplest family of all the curved surfaces, they form the basis for several solid-modelling systems[1,2].

When processing CAD models composed of quadric surfaces, it is necessary to compute their mutual curve of intersection. The algebraic analysis provides answers to the following queries:

- Do the surfaces actually intersect?
- Can the curve be reduced to components?
- Is the intersection planar?
- How can the curve(s) be parameterized?
- What are the equations of the curve(s) when it is projected onto a plane?

The answers to these queries can be used in rendering the intersection in a way that is faster than ray tracing.

ELOP Electro-Optics Industries, POB 1165, Rehovot 76110, Israel
*Israel Aircraft Industries — MALAT, Ben-Gurion International Airport, 70100, Israel
Paper received: 26 April 1990. Revised: 7 January 1993

Following the algebraic part, structural analysis is aimed at building a graph description of the intersection such that each arc of the graph corresponds to a part of a noncomposite curve. Such a description forms the basis for the boundary evaluation of the CAD model.

Quadric surfaces are conveniently represented in the matrix form:

$$\mathbf{pQp}^t = 0 \tag{1}$$

where $\mathbf{p} = [x \ y \ z \ 1]$, and $\mathbf{Q}$ is a 4×4 symmetric discriminant matrix.

Given two such quadrics, we present here a method for the geometrical and structural analysis of their intersection. Only real, nonplanar intersections are discussed, since the planar cases are handled effectively elsewhere[3,4].

Intersection curve description consists of four consecutive phases:

- *Step 1:* classification of the intersection by means of invariant factors,
- *Step 2:* computing the geometric (algebraic) form of intersection,
- *Step 3:* determining the structure of the intersection curve,
- *Step 4:* analysing the projection of the curve.

For the sake of completeness, we begin with the required mathematical background, in the following two sections.

CLASSIFICATION BY INVARIANT FACTORS

To analyse the intersection of a pair of quadrics Q_1 and Q_2, it is sufficient[5] to consider any two distinct quadrics of the parametric family (also known as a *pencil of quadrics*):

$$\mathbf{C} \equiv \mathbf{Q}_1 - \lambda \mathbf{Q}_2 \tag{2}$$

0010–4485/93/100633–11 © 1993 Butterworth-Heinemann Ltd

where λ is a parameter. The analysis is greatly simplified if one of these quadrics is singular, that is if λ satisfies the *characteristic equation*

$$|\mathbf{Q}_1 - \lambda \mathbf{Q}_2| = 0 \tag{3}$$

If the left-hand side of Equation 3 is identically zero, then the pencil is singular. Otherwise, we obtain a quartic equation

$$\Delta_2 \lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + \Delta_1 = 0 \tag{4}$$

where Δ_1 and Δ_2 are the discriminants of Q_1 and Q_2, respectively. The equation has four roots (one of them at infinity if $\Delta_2 = 0$). Hence, in a nonsingular pencil of quadrics, there are four distinct or coincident singular surfaces (possibly with complex coefficients).

The geometric part of the method presented here is a modification of Levin's method[5], guided by invariant-factors pencil classification[6,7]: considering the complex roots of Equation 4, it has been shown that certain properties of these roots are sufficient to characterize the pencil both algebraically and topologically.

Consider a root c of Equation 4. Let $(\lambda - c)^{l_0}$ be a factor of the determinant of Equation 2. Let $(\lambda - c)^{l_1}$ be a factor of the determinants of all its first minors. Define $(\lambda - c)l_2$ similarly for the second minors. . . . Let l_r be the first index in the sequence which is zero.

Define a sequence of indices d_i by the successive differences

$$d_1 = l_0 - l_1,\ d_2 = l_1 - l_2,\ \ldots,\ d_r = l_{r-1}$$

The powers $(\lambda - c)^{d_1}$, $(\lambda - c)^{d_1}$, . . . $(\lambda - c)^{d_r}$ of $(\lambda - c)$ are called *invariant factors* to the base $(\lambda - c)$ of the determinant of the pencil. The *Segre characteristic* is the string obtained by putting the exponents d_i associated with the same root in parentheses, and all the sets for all the bases in square brackets.

The nonplanar pencils are listed in *Table 1* (see Reference 6, pp 46, 47). They can be reduced to a canonical representation (for both quadrics) whose form depends on the invariant factors only. The reduction is carried out by means of a projective coordinate transformation. In the reduced form, the QSIC (quadric-surface intersection curve) can be analysed, almost by inspection. However, the reduction is rather involved[8]. By the adaptation of Levin's method, such a complicated reduction is no longer necessary.

Table 1 Segre characteristic strings and corresponding forms of nonplanar quadric-surface intersection curves

Characteristic	Form of QSIC
[1111]	Nonsingular-quartic
[112]	Nodal-quartic
[13]	Cuspidal-quartic
[22]	Space-cubic and bisecant
[(11)(11)]	Four generators
[4]	Space-cubic and tangent

In the following sections, we show that the type of intersection (the Segre characteristic) can be readily derived by computing all the roots of a quartic or a cubic equation. Thus we can avoid computing the Segre characteristic by following its rather complex definition.

LEVIN'S METHOD

Levin has observed that the real roots of Equation 4 yield ruled surfaces which can be effectively intersected with another quadric of the pencil. If all four roots are complex, a hyperbolic paraboloid can be found among

$$\mathbf{Q}_1 - \lambda_i \mathbf{Q}_2 = \mathbf{0}$$

where λ_i is a root of the cubic characteristic polynomial associated with the 3×3 upper-left subdiscriminants of Q_1 and Q_2. For real roots and nonplanar intersections, a cone or a cylinder corresponds to the root of highest multiplicity.

The ruled surface is reduced, by a series of transformations, to an appropriate normal form, as given in *Table 2*.

In normal form, the ruled surfaces are easily described by a family of generators (see *Table 2*): a parameter t selects a generator, and s controls the movement along the generator. The parameterization of points (x, y, z, w) on the ruled surface is described in *Table 2*. Substituting this parameterization into the explicit form of a different quadric in the pencil yields the generator-QSIC equation of intersection:

$$A(t)s^2 + B(t)s + C(t) = 0 \tag{5}$$

The respective degrees of the A, B, C polynomials are also given in *Table 2*.

Our description of the method is slightly different from the original one in the following ways:

- The real root of highest multiplicity has been chosen to obtain the simplest quadric form to be reduced.
- We have used homogeneous coordinates, to consider generators at infinity.
- An alternative family of generators lying on the hyperbolic paraboloid has been included. It is used below to decompose nonplanar intersections.

Levin's algorithm for QSIC tracing is as follows:

(1) Consider a closed spaced sequence of t values (each corresponding to a generator).
(2) For each t:
  (2a) Substitute t values in Equation 5.
  (2b) Solve the (generally) quadratic equa- for s.
  (2c) For every (t, s) pair compute the corresponding point on the curve.
  (2d) Transform the points into the view space and display.

Table 2 Nonplanar QSIC parameterizations

	Cone	Cylinders			Hyperbolic paraboloid	
		Elliptic	Parabolic	Hyperbolic	Family 1	Family 2
Normal	$xy=z^2w$	$x^2+z^2=w^2$	$x^2=zw$	$xz=w^2$	$xy=zw$	
x	s	$\frac{2t}{1+t^2}$	t	t	t	s
y	st^2	$\frac{s}{1+t^2}$	s	s/t	s	t
z	ts	$\frac{1-t^2}{1+t^2}$	t^2	$1/t$	ts	ts
w	1	1	1	1	1	1
Degrees of A, B, C in Equation 5	4, 2, 0	0, 2, 4	0, 2, 4	0, 2, 4	2, 2, 2	2, 2, 2

[For each ruled surface of parameterization, a normal canonical form is provided. In that form, the points (x, y, z, w) on the surface are described by two parameters: t selects a generator, while s controls movement along the generator.]

Figure 1 shows two hyperboloids of one sheet such that their intersection decomposes into a generator and a space cubic (a pencil of characteristic [22]). By applying Levin's method to this arrangement (see *Figure 2*), the cubic is traced correctly. However, the generator is missing. We show a correct result further below, computed by the algorithms described in this paper.

CLASSIFICATION OF QSIC

By inspecting *Table 1*, we conclude that all the multiple roots of Equation 4 must be real, except for the [22] case, where the roots may consist of two identical complex conjugate pairs. In such a case, the only possible ruled surface in the system is a hyperbolic paraboloid. This case is handled separately later.

For all real, multiple roots of interest, the powers l_i, $i>0$, defined above must equal zero, or the pencil will be planar. Hence the multiplicity of the real roots serves as an index to *Table 1*, and classification is trivial.

GEOMETRIC CURVE DESCRIPTION

Geometrically, a nonplanar QSIC is a list of real irreducible components of the following types: line, nonsingular-quartic, nodal-quartic, cuspidal quartic and cubic. All the types can be parameterized (rationally, except for the nonsingular-quartic type). The parametric forms of the curves are generally given in some *parameterization space*, where the equations take a particularly simple form.

Parametric forms of singular quartics

The singular quartics in the family of QSICs consist of the nodal quartic (characteristic [112]) and the cuspidal quartic ([13]). The multiple root of Equation 3 is real, and the homogeneous coordinates of points on the curve can be represented by polynomials. The ruled surface computed above is a cone or a cylinder.

The parameterization on a cone becomes a rational one if $C(t)$ (Equation 5) vanishes. For a cone, $C(t)$ is the 4, 4 entry of the discriminant matrix of O: another quadric of the pencil. Therefore, the vertex of the cone, at the origin of parameterization space, lies on every quadric of the system. The quadratic equation

$$A(t)s^2+B(t)s=0$$

decomposes into

$$A(t)s+B(t)=0$$

where $s=0$.

The second component corresponds to the cone's vertex. Substituting

$$s=-B(t)/A(t)$$

from the first component into the forms in *Table 2*, we obtain the rational form in *Table 3*.

For cylinders, $A(t)$ is a constant. If it vanishes, s can be expressed by $-C(t)/B(t)$, and a rational parametric form is obtained (see *Table 3*).

Line-cubic composites on singular quadrics

The pencils of quadrics of types [4] and [22] (with real roots) intersect in a line-cubic composite curve (see *Table 1*). The system contains a singular nonplanar quadric (a cone or a cylinder).

The A, B, C polynomials satisfy the rationality conditions stated above: A vanishes for cylinders, and C for cones. The generator accompanying the cubic is

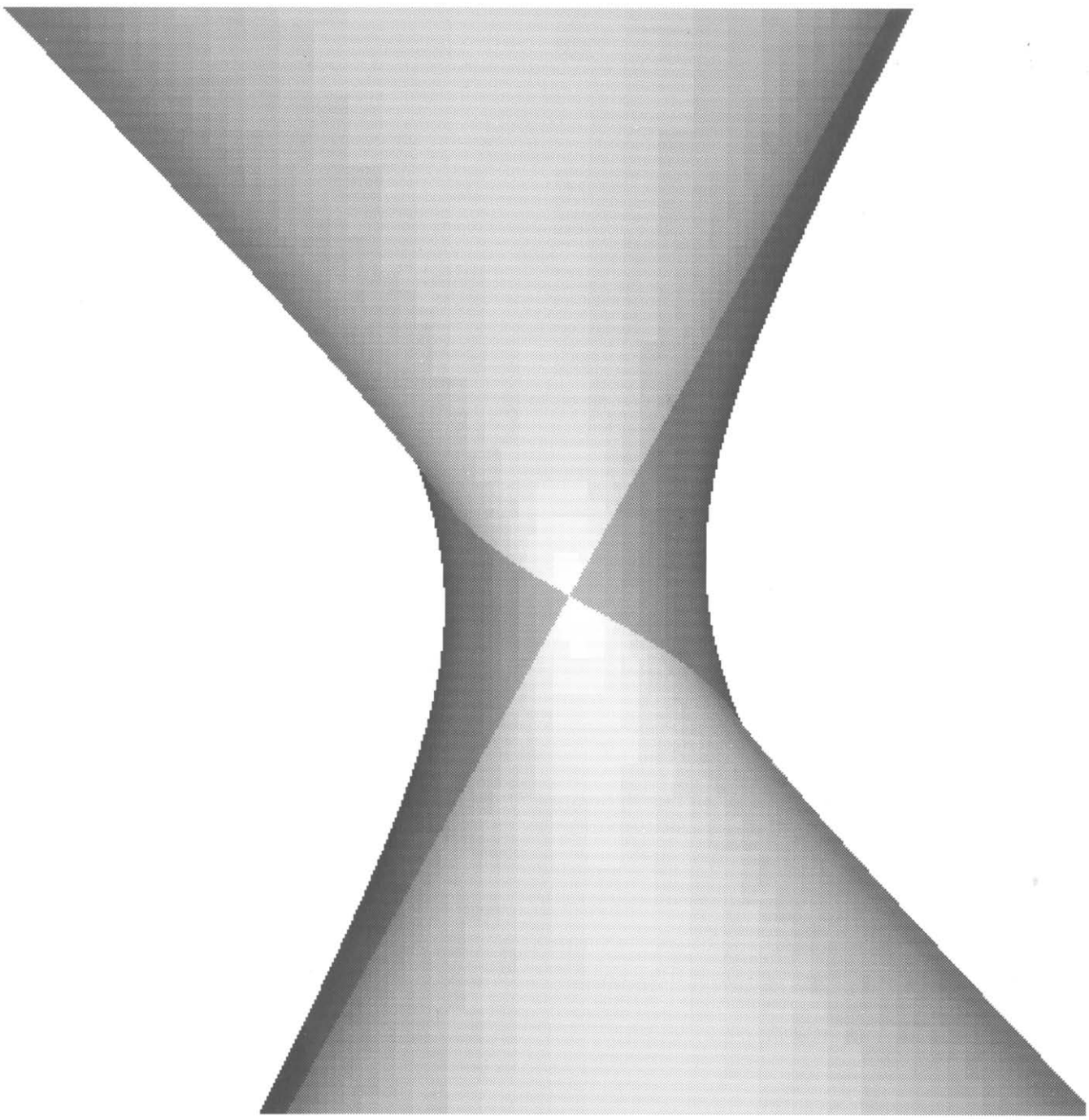


Figure 1 Ray-traced image showing two hyperboloids of one sheet [The intersection decomposes into a generator and a space cubic.]

related to the zeros of these polynomials. Finding the generator, the polynomials are reduced, providing a rational parameterization for the cubic component. The following derivation shows the incompleteness of Levin's method whenever generators in nonplanar systems are involved.

Consider the equation

$$A(t)s^2 + B(t)s + C(t) = 0$$

in which t selects the generator, and s controls motion along the generator. For some generator t_0 to be a part of the QSIC, it must lie on the other surface O. Hence,

$$A(t_0) = B(t_0) = C(t_0) = 0 \tag{6}$$

For a cone, C vanishes. Dividing both A and B by $(t - t_0)$, A is now of order 1 and B of order 3. Substituting A, B in the rational form for a QSIC on the cone, a rational cubic is obtained. For cylinders, we reduce B and C in a similar manner. Note that, for a generator corresponding to an infinite parameter value (such as the $x = 0$, $y = -1$ generator on the elliptic cylinder), the polynomials are already reduced.

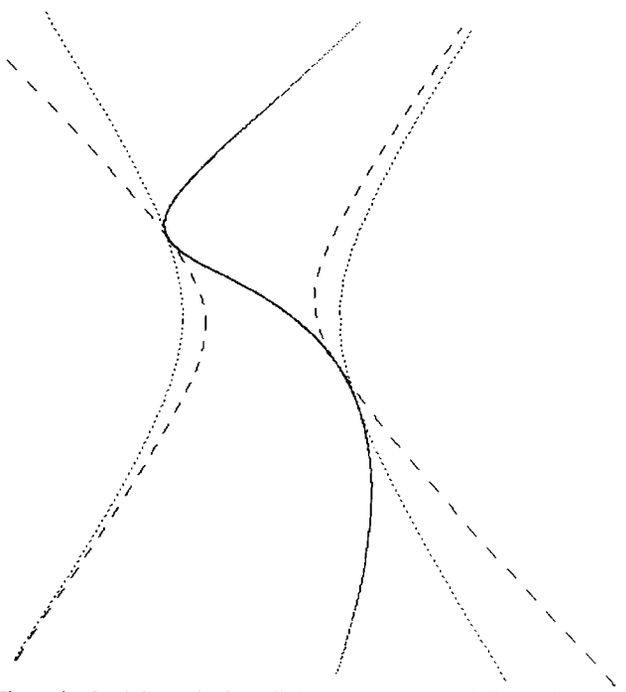


Figure 2 Levin's method applied to arrangement of *Figure 1*
[The silhouette curves of the quadrics are shown by broken or dotted lines. The cubic (shown by a solid line) is traced correctly. However, the generator is missing.]

Table 3 Rational parameterization on singular irreducible quadrics

	Cone	Cylinders	
		Elliptic	Parabolic	Hyperbolic
Rationality condition	$O[4][4]=0$		$O[2][2]=0$	
Components	$s=0$ (vertex)		$s=\frac{-C(t)}{B(t)}$	
	$s=\frac{-B(t)}{A(t)}$			
x	B		tB	t^2B
y	t^2B		$-C$	$-C$
z	tB		t^2B	B
w	$-A$		B	tB

[The rationality conditions on the other quadric's coefficients are displayed, with the rational form for s and for the coordinates on the curve. All the rational forms are in terms of the coefficients in Equation 5.]

Another exception involves the generator at infinity, which may be the generator of the system. Although invisible, it is algebraically a component of the intersection, and it should be kept for the factorization of the projected quartic (see below).

We summarize line-cubic decomposition for singular surfaces as follows:

- Look for a generator at infinity or a generator corresponding to $t=\infty$. If found, return (polynomials already reduced).
- Find the common root t_0 of the polynomials. (There must be one, for a real intersection).
- Reduce the polynomials to obtain the rational representation of the cubic.
- Compute the generator described by $t=t_0$ (see *Table 4*).

Details specific to various parameterization surfaces are described in *Table 4*. *Table 4* includes the relevant formulae for the hyperbolic paraboloid, to be discussed next.

Nonplanar QSIC on hyperbolic paraboloid

Suppose that the characteristic equation yields two pairs of identical complex roots. In such a case, the parameterization surface is a hyperbolic paraboloid. Looking for appropriate nonplanar systems in *Table 1*, we locate [22] (cubic and bisecant) and [(11)(11)] (four generators). The distinction between the two types is made by finding all the generators of the system. If only one such generator exists, the system is of [22] type, and the second component is a rational cubic. Otherwise, the characteristic is [(11)(11)].

The generators are detected using the generic line-cubic decomposition algorithm. It is applied to both families of generators in *Table 2*. The additional data required by the algorithm are given in *Table 4*. If no line has been found, the quadrics do not intersect.

Table 4 Line-cubic decomposition: auxiliary table

	Cone	Cylinders			Hyperbolic paraboloid	
		Elliptic	Parabolic	Hyperbolic	Family 1	Family 2
Normal form	$xy=z^2w$	$x^2+z^2=w^2$	$x^2=zw$	$xz=w^2$	$xy=zw$	
Generator at infinity		None	$x=0$ $w=0$	$x=w=0$ $z=w=0$	$x=0$ $w=0$	$y=0$ $w=0$
Generator corresponding to $t=\infty$	$x=0$ $z=0$	$x=0$ $y=-1$				
Generator corresponding to t	$\frac{z}{x}=t$	$y=\frac{1-t^2}{1+t^2}$	$x-t=0$	$x-t=0$	$\frac{z}{y}-t=0$	$y-t=0$
	$\frac{y}{x}=t^2$	$x=\frac{2t}{1+t^2}$	$z-t^2=0$	$z-\frac{1}{t}=0$	$x-t=0$	$\frac{z}{x}-t=0$

[Enables it to be checked whether a special generator is a part of the intersection, and the general-case generator to be computed otherwise.]

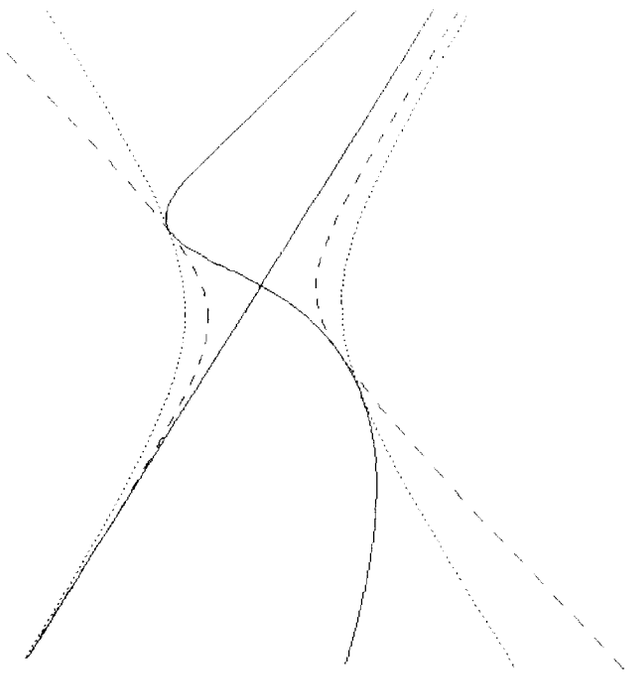


Figure 3 Complete intersection (shown by solid line) of quadrics in *Figure 1* as computed by our algorithm
[The silhouette curves are shown as in *Figure 2*.]

Levin's method revisited

For nonplanar systems, containing generators, Levin's approach fails, even for tracing purposes. The A, B, C polynomials evaluated as the coefficients of the quadratic equation have a common factor corresponding to the generator. That common factor cancels out in the solution of the equation, and therefore only the cubic is being traced. The same holds for multiple generators on the hyperbolic paraboloid.

Using the rational form of the cubic and the line, separately, the full intersection of the pair of quadrics can be traced correctly. The complete intersection of the arrangement in *Figure 1* is shown in *Figure 3*.

STRUCTURE OF INTERSECTION CURVE

The curve segments of the intersection curve are *labelled arcs* of the curve graph. Each such arc is described by a quadruple of the form

$$(nc, sgn, t_1, t_2)$$

where nc is a noncomposite curve descriptor, sgn is a sign variable (for a nonsingular quartic, $sgn \in \{-1, 1\}$ selects one solution of the quadratic in Equation 5; otherwise, $sgn = 0$), and t_1 and t_2 define the corresponding parametric interval.

The nodes of the graph, which determine the spatial structure of the curve, include the following:

- a node at infinity,
- an actual double point,
- a 'dummy' node, connecting two contiguous arcs with a different parameterization; either $-\infty$ meets ∞, or each of the two arcs has a different Boolean sign entry.

Graph of nonsingular quartics

Given the A, B, C algebraic representation of a nonsingular quartic, qt, the real parameter axis $(-\infty, \infty)$ is segmented by zeros of the quadratic discriminant

$$D \equiv A^2 - 4BC$$

For each of the intervals obtained, say (t_1, t_2), we evaluate D in some $t_0 \in (t_1, t_2)$ such that $A(t_0) \neq 0$. If $D(t_0) > 0$, then the 2-sided (positive and negative sign) interval (t_1, t_2) contains real points, and the following arcs are appended to the graph:

- $(qt, 1, t_1, t_2)$
- $(qt, -1, t_1, t_2)$

We complete the 3D graph construction by adding a node at infinity at the zeros of A both for the positive and the negative root sign.

Figure 4 shows two intersecting ellipsoids. The nonsingular quartic of intersection consists of two branches. The parameterization surface is an elliptic cylinder. The corresponding computed intersection is shown in *Figure 5*.

Graph of rational quartics

The parameterization domain for the rational quartics consists of the entire real axis. For a cone, the zeros of B which correspond to the vertex are added as nodes (these zeros cannot cancel with those of A, or the curve will be reducible). The graph is constructed as follows:

- Make a sorted parameter list composed of $-\infty$, ∞ (plus the zeros of B if the ruled surface is a cone).
- For each of the intervals defined by pairs of adjacent parameter values (t_1, t_2), form the arc $(qt, 0, t_1, t_2)$.
- Mark each node of the graph which coincides with the cone's vertex as a double point.
- For every zero of w, add a node at infinity.

Figure 6 shows an ellipsoid intersecting with a hyperboloid of one sheet. The corresponding computed nodal quartic is shown in *Figure 7*. The quartic (shown

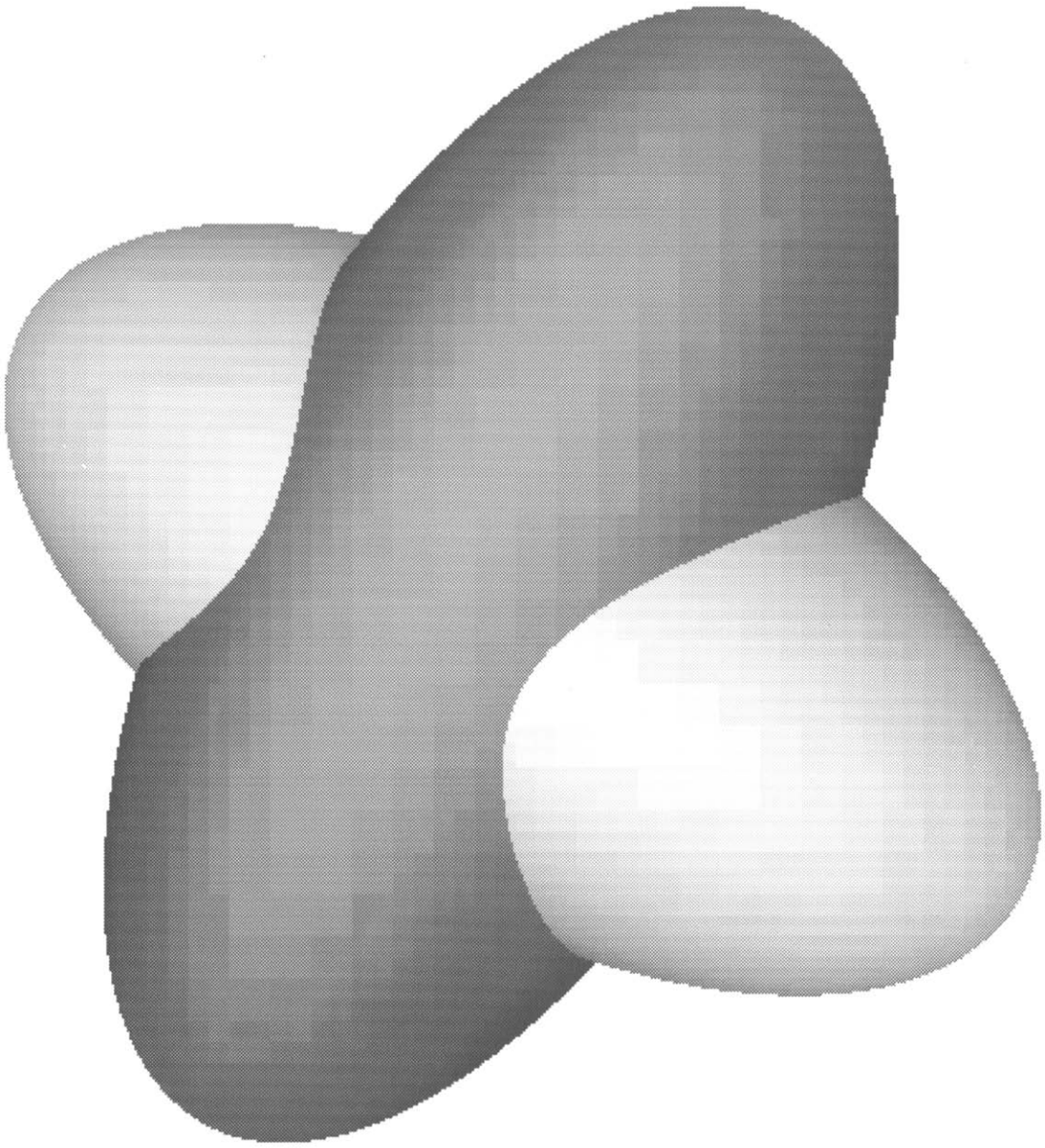


Figure 4 Ray-traced image showing two intersecting ellipsoids [The intersection consists of two branches.]

by a solid line) is composed of the arcs $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$. The node N corresponds to the parameter values ± 1. D is the 'dummy' node at $\pm\infty$.

Line–cubic graph construction

Cubic components of the QSIC are rational. Therefore, the structure of the cubic component can be derived in a similar way to that for the construction of the rational quartics. The generator, if not at infinity, is properly added to the graph. The construction is summarized below:

- Let *cubic* be the algebraic cubic descriptor.
- Initialize the graph to the arc (*cubic*, 0, $-\infty$, ∞).
- Add a node at infinity, to the cubic, at all zeros of the w coordinate polynomial.
- If the associated generator is at infinity, return.
- Let *line* be the generator.
- Add the arc (*line*, 0, $-\infty$, ∞) to the graph.
- Add all the points common to the cubic and to the line as double points.

The intersection between the cubic and the associated generator is obtained by solving for the intersection between the cubic and one of the line's planes, using the rational form of the cubic. Only the intersection points lying on the second plane are considered. In addition, if the ruled surface is a cone whose vertex corresponds to $t=\infty$, it is added to these points. The last check is accomplished by testing whether

$$\deg(A)-\deg(B)>2$$

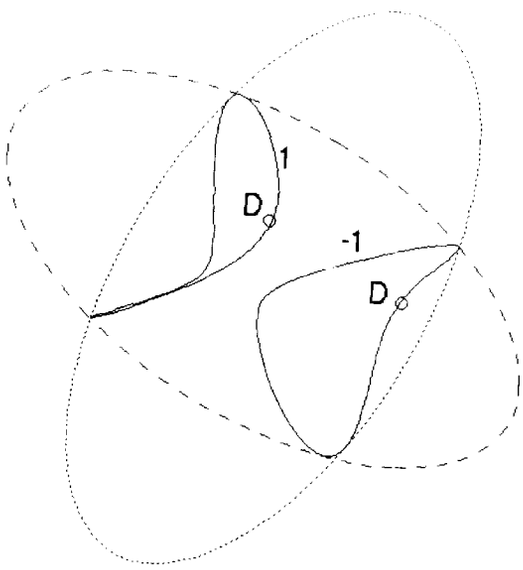


Figure 5 2-branch nonsingular quartic (shown by solid line) of intersection for arrangement shown in *Figure 4*
[The silhouette curves of the quadrics are shown by broken or dotted lines. 'Dummy' nodes associated with discontinuity of parameter values are marked *D*. The branches can be distinguished by the quadratic root sign (either −1 or 1).]

For multiple generators on a hyperbolic paraboloid, a graph of a line is constructed for each of the components. Owing to the nonplanarity assumption, the generators do not intersect.

PROJECTING QSIC GRAPHS

The projection of the intersection curve (on the xy plane, by convention) is useful both for visualization and for boundary evaluation[9].

Let the implicit forms of both the quadrics in homogeneous coordinates be given by

$$q(x, y, z, w)=0$$

$$Q(x, y, z, w)=0$$

Representing the quadratics as polynomials in z,

$$\begin{aligned} az^2+bz+c&=0 \\ Az^2+Bz+C&=0 \end{aligned} \tag{7}$$

the algebraic plane curve of intersection can be obtained by eliminating z between the equations. For line–cubic composites, the curve obtained by elimination is reduced, pseudodividing it by the generator's equation.

The projection generates additional nodes which include the following:

- 'Contact points'[10], where the curve touches the silhouette curve of one of the associated quadrics. At such points, the curve passes from the front side of the quadric (where the z component of the surface normal points in the direction of projection) to its back, or *vice versa*.
- Extrema and inflection points of the projected curve. The proper identification of these points is essential for the efficient sequential following of curves on raster displays[11].
- Apparent double points where the projected curve intersects itself. Using the representation in Equation 7, it can be shown[12] that these points are defined by the intersection of the conic

$$cA-Ca=0$$

with the line

$$bA-Ba=0$$

(see *Figure 8*).

To associate a depth value with these points, the common roots of Equations 7 must be found.

EXAMPLE

Figures 1 and *3* show a pair of intersecting quadrics. We trace the analysis of the intersection, through the algorithms described in this paper.

Step 1: Classification

(1.1) The characteristic Equation 3 translates, by substitution, to

$$\lambda^4-16\lambda^3+94\lambda^2-240\lambda+225=0$$

(1.2) Solving the equation, we obtain a double root at $\lambda=5$, and a double root at $\lambda=3$.

(1.3) The intersection is not planar (this can be verified at the first stage of analysis, by testing for planarity[5]).

(1.4) Since we have a pair of double real roots, we conclude from *Table 1* that the Segre characteristic is [22].

Step 2: Geometric description

(2.1) Taking $\lambda=5.0$, we obtain the ruled surface $\mathbf{Q}_1-5\mathbf{Q}_2$.

(2.2) The ruled surface is classified as a parabolic cylinder.

(2.3) The parabolic cylinder is reduced to the normal form described in *Table 2*.

(2.4) The same transformation sequence is applied to another quadric of the family (say $\mathbf{Q}_1$).

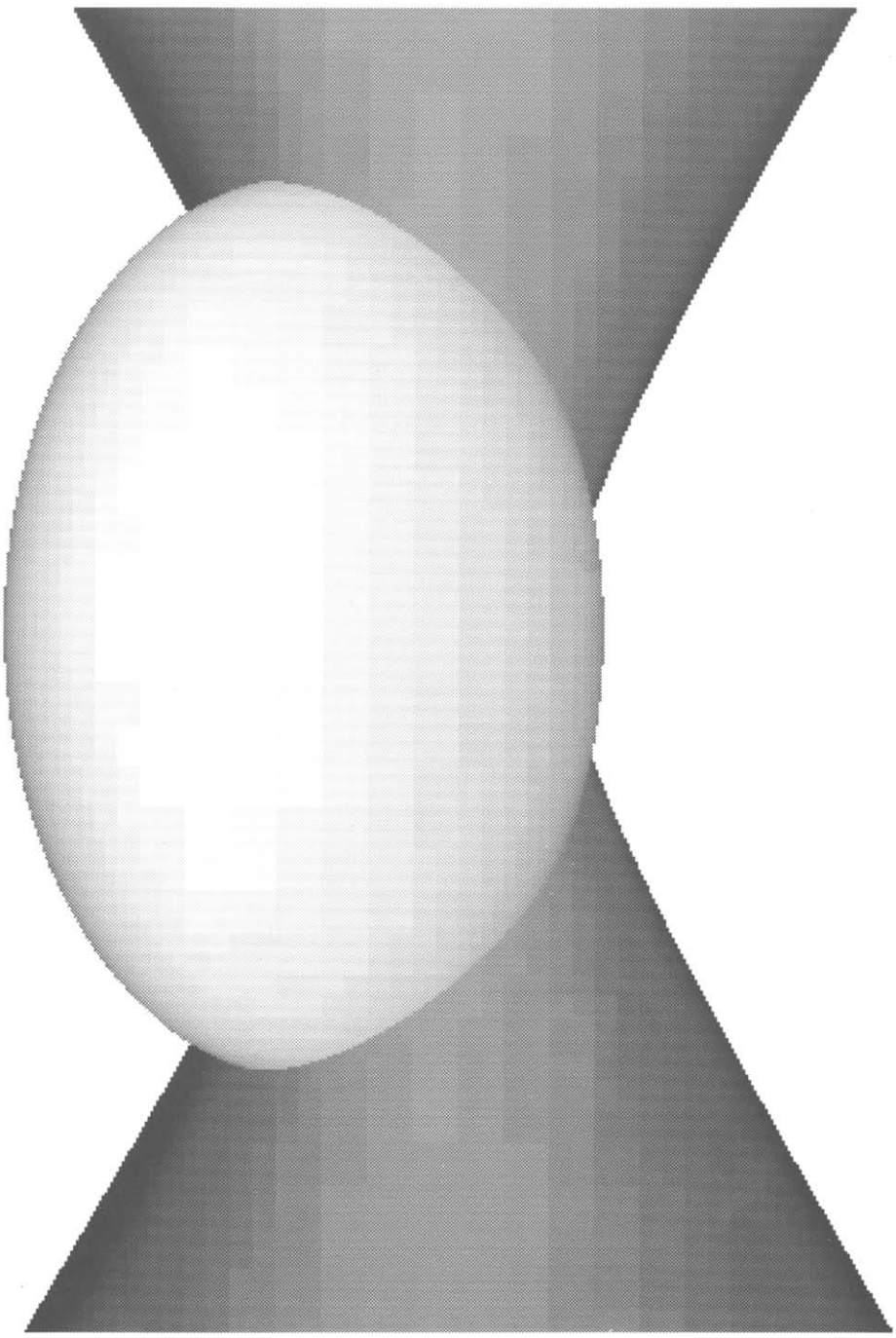


Figure 6 Ray-traced image showing ellipsoid intersecting with hyperboloid of one sheet

(2.5) Substituting the parametric form of generators on the parabolic cylinder in the transformed quadric $\mathbf{Q}_1$, we obtain an equation of the form of Equation 5, where

$$A(t)=0$$

$$B(t)=-10t$$

$$C(t)=1.25t^4+13.92t$$

(2.6) Substituting $x=w=0$ in the transformed quadric form associated with $\mathbf{Q}_1$, it does not vanish. Therefore, the generator $x=w=0$ does not lie on $\mathbf{Q}_1$.

(2.7) We proceed by finding all the common roots of $A(t)$, $B(t)$, $C(t)$. Clearly, they have only $t_0=0$ in common.

(2.8) From the last row in the parabolic-cylinder column (see *Table 4*) the bisecant is given (in the coordinate space where the cylinder has its normal form) by

$$x=z=0$$

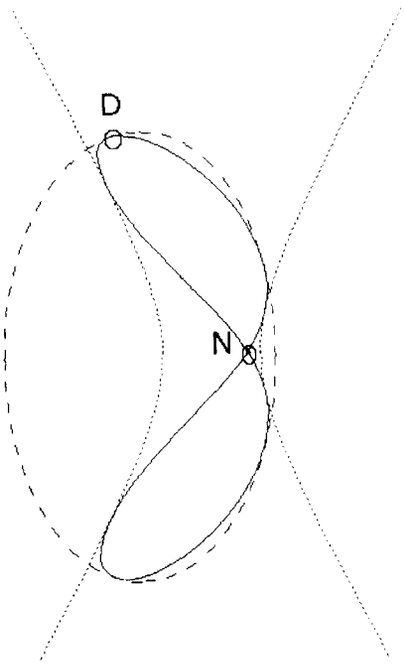


Figure 7 Computed intersection of arrangement in *Figure 6*
[The quartic (shown by a solid line) is a nodal one (N). A 'dummy' node associated with a discontinuity of parameter values is marked D. The silhouette curves of the quadrics are shown by broken or dotted lines.]

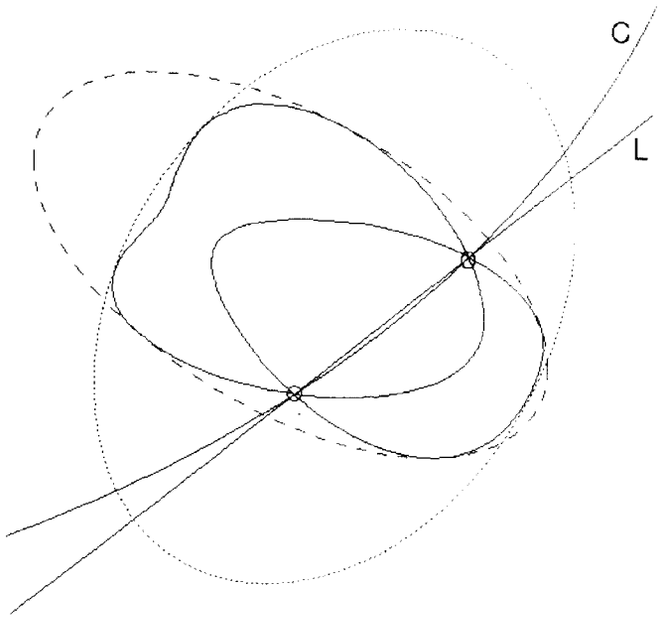


Figure 8 The apparent double points of a quartic are those obtained by self intersection of the curve when projected onto a plane
[These points can be obtained as the intersection of a computed line L and a conic C, both lying on the projection plane. The silhouette curves of the quadrics are shown by broken or dotted lines.]

Divide the A, B, C polynomials by the factor t to obtain the rational parametric form of the cubic (see *Table 3*):

$$x = tB$$
$$y = -C$$
$$z = t^2 B$$
$$w = tB$$

where

$$B(t) = -10$$
$$C(t) = 1.25t^3 + 13.92$$

Step 3: 3D structure

(3.1) The graph is initialized to a pair of arcs: one for the cubic and one for the line.

(3.2) The point common to the cubic and the line is computed as described above.

Step 4: Projection

(4.1) Eliminating z in Equation 7 results in the following plane quartic:

$$\begin{aligned}0.118x^4 + 0.518x^3z + x^2z^2 + 0.715xz^3 + 0.165z^4 \\ + 0.408x^3 + 0.781x^2z + 0.358xz^2 \\ + 0.008z^3 + 0.409x^2 - 0.009xz - 0.224z^2 \\ + 0.078z - 0.090z - 0.001 = 0\end{aligned}$$

(4.2) Transforming the $x=0$, $z=0$ plane back to the viewing coordinate system and eliminating, we obtain the projected generator

$$x + 0.618z + 0.273 = 0$$

(4.3) Pseudodividing the quartic by the line and normalizing, we obtain, for the plane cubic,

$$\begin{aligned}0.163x^3 + 0.613x^2z + xz^2 + 0.368z^3 + 0.517x^2 \\ + 0.590xz - 0.144z^2 + 0.422x - 0.435z \\ - 0.007 = 0\end{aligned}$$

DISCUSSION

A method for deriving the shape and structure of quadric-surface intersection curves has been presented. Only nonplanar intersections have been discussed, although the method handles planar intersections as well[12]. Efficient computational algorithms for deriving the geometric and topological properties of the curves have been described.

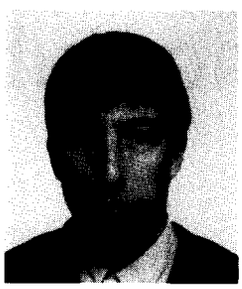
Directed by the Segre characteristic of the pencil of quadrics, Levin's method has been extended and corrected to provide rational parameterizations and cubic–line decomposition. The Segre characteristic and the parameterization polynomials determine the 3D connectedness of the intersection graph.

The algebraic nature of the method, as opposed to geometric methods[13,14], ensures its generality. Recently, an equivalent algebraic method for QSIC shape analysis

has been presented[15]. That method requires the multivariate factorization of the projecting cone of the quartic curve. Therefore, it is restricted to quadrics with rational coefficients. Owing to the concept of ruled-surface parameterization, our method tends to be simpler. It also facilitates the QSIC graph construction not addressed by Reference 15.

REFERENCES

1 Boyse, J W and Gilchrist, J E 'GMSolid: interactive modeling for design and analysis of solids' *IEEE Comp. Graph. Applic.* Vol 2 (Mar 1982)
2 Brown, C M 'Padl-2: a technical summary' *IEEE Comp. Graph. Applic.* Vol 3 (Mar 1982) pp 69–84
3 Levin, J Z 'Mathematical models for determining the intersections of quadric surfaces' *Comput. Graph. Image Proc.* Vol 11 (1979) pp 73–87
4 Sarraga, R F 'Algebraic methods for intersections of quadric surfaces in GMSOLID' *Comput. Vision, Graph. Image Proc.* Vol 22 (1983) pp 222–238
5 Levin, J Z 'A parametric algorithm for drawing pictures of solid object composed of quadric surfaces' *Commun. ACM* Vol 19 (1976) pp 555–563
6 Bromwich, T J *Quadratic Forms and Their Classification by Means of Invariant Factors* Hafner, USA (1906)
7 Salmon, G *Analytic Geometry of Three Dimensions* Chelsea (1927)
8 Snyder, V and Sisam, C H *Analytic Geometry of Space* Henry Holt (1925)
9 Wilf, I and Manor, Y 'Splitting quadric faces' *Comput.-Aided Des.* Vol 22 (1990) pp 507–514
10 Gardner, G Y 'Simulation of natural scenes using textured quadric surfaces' *Comput. Graph.* Vol 18 (1984) pp 11–20 (*Proc. ACM/SIGGRAPH Conf.*)
11 Wilf, I and Manor, Y 'Tracking parameterized, algebraic curves on raster displays' *Comput. Graph.* Vol 13 (1989) pp 355–359
12 Wilf, I 'Logical manipulation of three-dimensional geometry' *PhD Thesis* Fac. Engineering, Tel-Aviv University, Israel (Aug 1989)
13 Miller, J R 'Geometric approaches to nonplanar quadric surface intersection curves' *ACM Trans. Graph.* Vol 6 (1987) pp 274–307
14 Miller, J R 'Analysis of quadric-surface-based solid models' *IEEE Comp. Graph. Applic.* (Jan 1988) pp 28–42
15 Farouki, R T *et al.* 'Automatic parsing of degenerate quadric-surface intersections' *ACM Trans. Graph.* Vol 8 (1989) pp 174–203



Itzhak Wilf received a BSc in electrical engineering from the Technion in Haifa, Israel, in 1979. He received an MSc and a PhD in electrical engineering from Tel-Aviv University, Israel, in 1983 and 1990. His career includes military service as an engineer as well as research and consulting for Israeli industrial firms. From 1989 to 1992, he was the chief engineer for image processing and algorithms at Optrotech, Israel. He is currently the head of the image processing group at ELOP Electro-Optics Industries.

Yehuda Manor received a PhD from the Hebrew University, Jerusalem, Israel, in 1976, and has held positions in Cambridge, UK, and in Stanford, USA. He became a full professor in 1983. In conjunction with his academic activities, he has been involved in R&D industrial projects in GE, Israel Aircraft Industries, and other major companies.